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Geodetic Graphs and Convexity.

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GEODETIC GRAPHS AND CONVEXITY

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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Abstract

A graph is geodetic if each two vertices are joined by a unique shortest path. The problem of characterizing such graphs was posed by Ore in 1962; although the geodetic graphs of diameter two have been described and classified by Stemple and Kantor, little is known of the structure of geodetic graphs in general. In this work, geodetic graphs are studied in the context of convexity in graphs: for a suitable family Π of paths in a graph G , an induced subgraph H of G is defined to be Π -convex if the vertex-set of H includes all vertices of G lying on paths in Π joining two vertices of H . Then G is Π -geodetic if each Π -convex hull of two vertices is a path. For the family Γ of geodesics (shortest paths) in G , the Γ -geodetic graphs are exactly the geodetic graphs of the original definition. For various families Π , the Π -geodetic graphs are characterized. The central results concern the family T of chordless paths of length no greater than the diameter; the T -geodetic graphs are called ultrageodetic. For graphs of diameter one or two, the ultrageodetic graphs are exactly the geodetic graphs. A geometry (P, L, F) consists of an arbitrary set P , an arbitrary set L , and a set $F \subseteq P \times L$. The point-flag graph of a geometry is defined here to be the graph with vertex-set $P \cup F$ whose edges are the pairs $\{p, (p, l)\}$ and $\{(p, l), (q, l)\}$ with $p, q \in P$, $l \in L$, and

$(p,1), (q,1) \in F$. With the aid of the Feit-Higman theorem on the nonexistence of generalized polygons and the collected results of Fuglister, Damerell-Georgiacodis, and Damerell on the nonexistence of Moore geometries, it is shown that two-connected ultrageodetic graphs of diameter greater than two are precisely the graphs obtained via the subdivision, with a constant number of new vertices, either of all of the edges incident with a single vertex in a complete graph, or of all edges of the form $\{p, (p,1)\}$ in the point-flag graph of a finite projective plane.

Chapter I. Notation, Terminology, and Historical Background

This report is couched in the language of graph theory; for terminology not defined here, the reader is referred to Bollobás [5]. We consider only finite simple graphs $G = (V, E)$; that is, the vertex-set $V = V(G)$ is a finite set, and the edge-set $E = E(G)$ is a subset of the collection $\binom{V}{2}$ of 2-subsets of V . All graphs are assumed to be connected.

In the main, we mention G explicitly only if it seems necessary to do so for the sake of clarity. In particular, we often use a single letter, say f , to denote one of a family of functions $\{f_G\}$ defined on the vertex-sets of various graphs G ; if more than one graph is under discussion, then we include the appropriate subscript.

In view of the fact that our discussion and proofs frequently involve the manipulation of paths and cycles, we present here an unusually detailed nomenclature for them. We understand a path P of length $\lambda(P) = k$ in a graph to be a sequence of $k+1$ distinct vertices, each consecutive two of which form an edge. If we need to show the sequence explicitly, then we write

$$(1) \quad P: x_0 - x_1 - \dots - x_k.$$

If P is a path with initial vertex x and terminal vertex y , then we call P an x, y -path, or we refer to $P(x, y)$; the vertices x and y are called the endpoints of P , and

all other vertices of P are called internal vertices. The converse of a path $P(x,y)$ is the path $P'(y,x)$ obtained via reversal of P . If v precedes w in a path P , then $P|v,w$ is the maximal subsequence of P beginning with v and ending with w ; it is called a subpath of P . When we refer to a family of paths, we mean a set Π of paths such that $P \in \Pi$ iff $P' \in \Pi$. We say that a family of paths Π connects G , or that G is Π -connected, if every two vertices of G are the endpoints of at least one path in Π . A family of paths Π is hereditary if it contains all of the subpaths of its members. Two paths P and Q are called internally disjoint if they have no vertices in common other than possibly their endpoints. If a vertex w is adjacent to the initial (resp., terminal) vertex of a path P , then we may prefix (resp., suffix) w to P to obtain the sequence wP (resp., Pw); it will be a path iff w is not a vertex of P . More generally, we may suffix a path $Q(y,z)$ to a path $P(x,y)$ by recursively suffixing the vertices of Q other than y (in order) to the sequence so far obtained from P to form a sequence PQ ; it will be a path iff P and Q have only the vertex y in common. If we suffix the initial vertex x of a path $P(x,y)$ to P , we obtain a cycle $C = Px$. For paths $P(x,y)$ and $Q(y,x)$, the sequences PQ and QP are cycles iff P and Q are internally disjoint. The length $\lambda(C)$ of a cycle C is just the number of its vertices. Vertices x and y in a cycle C are opposite one another in C if

each of the two x,y -paths in C has length no less than $\lfloor \lambda(C)/2 \rfloor$. If $\lambda(C)$ is even, then each vertex in C is opposite exactly one other; if $\lambda(C)$ is odd, then each vertex is opposite exactly two others.

By a subgraph, we mean an induced subgraph. If the subgraph induced by the vertices of a path P or cycle C has no edges other than those of the path or cycle, then we say that the path or cycle is chordless, and we refer to both the path or cycle and the subgraph induced by it with the same letter. If no two edges of the path or cycle lie in the same triangle, then we call the path or cycle locally chordless. If each of the internal vertices of a path has exactly two neighbors in the graph, and each of the endpoints has more than two neighbors, then the path is said to be suspended. For every path $P(x,y)$, each of whose endpoints has at least three neighbors, there is a unique decomposition of P into suspended subpaths $P = P_1 \dots P_n$, where $n - 1$ is the number of internal vertices of P that have at least three neighbors.

We can metrize V with the aid of the family Ω of all paths in G : for $x,y \in V$, the distance from x to y is

$$(2) \quad d(x,y) = d_G(x,y) = \min\{\lambda(P); P(x,y) \in \Omega\}.$$

Indeed, a metric can be obtained in this manner from any connected spanning subgraph. A path $P(x,y)$ of length $d(x,y)$ is called a geodesic. A subgraph H of G preserves distance in G , or is distance-preserving, if d_H

is the restriction of d_G to $V(H)$. The eccentricity of a vertex is

$$(3) \quad e(x) := \max\{d(x,y); y \in V\},$$

and the diameter of the graph is

$$(4) \quad \text{diam} = \text{diam}(G) := \max\{e(x); x \in V\}.$$

It is a simple matter to verify that d_G is a metric on $V(G)$; hence, we may define the sphere of radius $k \geq 0$ about a vertex x to be

$$(5) \quad S(x;k) := \{v \in V; d(v,x) = k\},$$

and the ball of radius k about x to be

$$(6) \quad B(x;k) := \{v \in V; d(v,x) \leq k\}.$$

The ball $B(x;1)$ and the sphere $S(x;1)$ are called the neighborhood of x and the punctured neighborhood of x , respectively. The number of vertices in the punctured neighborhood $S(x;1)$ is called the degree of x , and is denoted as $\delta(x)$. For $k \geq d(x,y)$, the k-reach of x through y is

$$(7) \quad R(x,y;k) := S(x;k) \cap B(y;k-d(x,y)),$$

and the reach of x through y is

$$R(x,y) := \bigcup \{R(x,y;k); d(x,y) \leq k \leq \text{diam}\}.$$

1.1 Remark. In fact, $R(x,y;k) = S(x;k) \cap S(y;k-d(x,y))$.

Moreover, the members of $R(x,y)$ are precisely the vertices z that satisfy the triangle equality

$$(8) \quad d(x,z) = d(x,y) + d(y,z).$$

We extend the preceding concepts to encompass subsets

A of V : The distance from A to a vertex x is

$$(9) \quad d(A,x) = d(x,A) := \min\{d(v,x); v \in A\}.$$

Accordingly, the sphere of radius $k \geq 0$ about A is

$$(10) \quad S(A;k) := \{x \in V; d(A,x) = k\},$$

and the ball of radius k about A is

$$(11) \quad B(A;k) := \{x \in V; d(A,x) \leq k\}.$$

The k -reach of A through a vertex x is

$$(12) \quad R(A,x;k) := S(A;k) \cap B(x;k-d(x,A)),$$

and the reach of A through x is

$$(13) \quad R(A,x) := \bigcup \{R(A,x;k); 0 \leq k \leq \text{diam}\}.$$

The cardinalities of the various sets just defined are denoted by the corresponding lower-case letters:

$$(14) \quad s(x;k) := |S(x;k)|, r(A,x) := |R(A,x)|, \text{ etc.}$$

1.2 Remark. As was the case with the reaches of vertices in 1.1, $R(A,x;k) = S(A;k) \cap S(x;d(A,x)-k)$, and the members of $R(A,x)$ are precisely the vertices y that satisfy the triangle equality

$$(15) \quad d(A,y) = d(A,x) + d(x,y).$$

A clique is a set of vertices that induces a complete subgraph, and a trivial clique is one containing two or fewer vertices.

In addition to the notation and terminology presented above, we shall apply several simple facts about graphs. The first one is a standard characterization of trees that forms the basis for the definition of geodetic graphs below:

1.3 Proposition. (see [29], IV 1) A graph is a tree iff each two vertices are joined by a unique path. ■

The intersection graph of a family of sets F is the graph whose vertex-set is F and whose edge-set consists precisely of the unordered pairs of elements of F with nonempty intersection. A block graph is the intersection graph of the vertex-sets of the blocks of some graph; the definition is somewhat unwieldy, so we shall use the following well-known characterization:

1.4 Proposition. [17] A graph is a block graph iff each of its blocks is complete. ■

The original impetus for this investigation was the following problem from Ore's 1962 monograph [33] on the Theory of Graphs:

"In a tree, there is a unique shortest arc between any two vertices, but there are also other connected graphs with the same property. Try to characterize these "geodetic" graphs in other ways." ([33], p. 104)

Although this problem has received considerable attention throughout the twenty-one years since it was posed, and despite the fact that several classes of examples have been constructed ([6], [10], [22], [30], [34], [35], [36]), very little is understood about the structure of geodetic graphs. One major difficulty is the fact that being geodetic is not an hereditary property; that is, a subgraph of a geodetic graph need not be geodetic: for each integer $n > 2$, the even cycle C_{2n} is not geodetic, but it is a subgraph of

the geodetic graph G_n of Figure 1.

That fact presents an obstacle to one standard method of graph theory, the "forbidden subgraph characterization". In that approach, one attempts to identify the set of forbidden subgraphs for a property P , that is, the graphs that do not enjoy property P , but whose every proper subgraph does enjoy P . If P is an hereditary property, then knowledge of the set of forbidden subgraphs for P yields a characterization for the graphs with that property: an arbitrary graph has property P iff it has no subgraph isomorphic to one of the forbidden subgraphs for P .

Even if the property in question is not hereditary, as is the case with geodetic graphs, there is a set of forbidden subgraphs, which may be empty. Although knowing which graphs belong to that set cannot provide a characterization, it does yield an answer to this question: given a graph G , can we construct a graph with the desired property that has a subgraph isomorphic to G ? To date, the only known forbidden subgraphs for geodetic graphs are the two that present themselves immediately: $K_{2,2}$ and K_4-e (see Figure 2). In fact, much of what is known of the structure of geodetic graphs follows from the fact that K_4-e is forbidden:

1.5 Proposition. [9] For a graph G , the following statements are equivalent:

- i) G has no subgraph isomorphic to K_4-e .
- ii) Each intersection of two maximal cliques of G

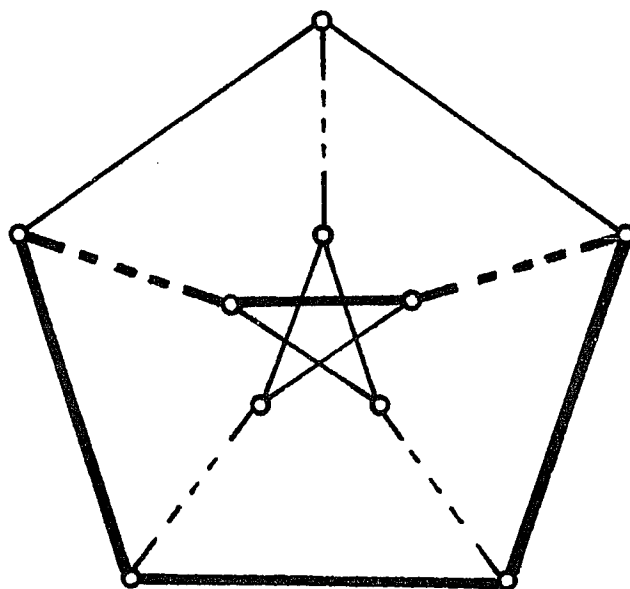


Figure 1.

The geodetic graph G_k , $k \geq 3$. Each broken line represents a suspended path of length $k - 1$. The heavy lines indicate a chordless cycle of length $2k$.

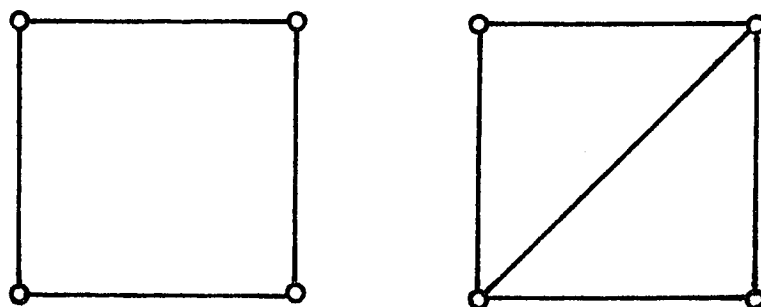


Figure 2.

Two forbidden subgraphs for geodetic graphs.

contains at most one vertex.

- iii) Each vertex outside a maximal clique has at most one neighbor in the clique.
- iv) Each edge lies in a unique maximal clique.
- v) For each pair of distinct vertices x, y in a maximal clique M , $R(M, x; 1) \cap R(M, y; 1) = \emptyset$.
- vi) For each vertex x , the punctured neighborhood $S(x; 1)$ is a disjoint union of cliques.

Proof. In each case, we prove the contrapositive of the implication in question.

i) \Rightarrow ii). Let M and N be distinct maximal cliques whose intersection contains the vertices x and y . Due to the maximality of M and N , there must exist nonadjacent vertices v and w in the symmetric difference $M \Delta N$. Now the subgraph induced by $\{v, w, x, y\}$ is isomorphic to $K_4 - e$.

ii) \Rightarrow iii). Suppose that a vertex x lying outside a maximal clique M has at least two neighbors in M , and let y and z be such neighbors of x . There is at least one maximal clique containing x, y , and z , say N . Since $x \notin M$, we know that $M \neq N$; however, $M \cap N$ contains both y and z .

iii) \Rightarrow iv). Suppose that the edge $\{x, y\}$ lies in two distinct maximal cliques M and N . Then there is a vertex $z \in M \setminus N$, and z has at least two neighbors in N , namely, x and y .

iv) \Rightarrow v). Suppose that there is a vertex w in

$R(M,x;1) \cap R(M,y;1)$ for some maximal clique M and distinct members x and y of M . Then the edge $\{x,y\}$ lies not only in the maximal clique M , but also in some maximal clique containing w .

$v) \Rightarrow vi)$. Suppose that there are vertices u , v , and w in $S(x;1)$ such that v is adjacent to u and w , but u and w are not adjacent. Let M be a maximal clique containing v , w , and x . Then u belongs to $R(M,v;1) \cap R(M,x;1)$.

$vi) \Rightarrow i)$. Suppose that G has a subgraph isomorphic to K_4-e with vertices v , w , x , and y such that v and w are not adjacent. Then no partition of $S(x;1)$ into cliques covers both of the edges $\{v,y\}$ and $\{w,y\}$. ■

Fortunately, there is one sense in which being geodetic is an hereditary property; as the following observation of Stemple and Watkins shows, we may restrict our attention to two-connected geodetic graphs:

1.6 Proposition. [42] A graph is geodetic iff each of its blocks is geodetic. ■

If a graph has diameter two, then it is easy to see that it is geodetic iff it has no subgraph isomorphic to $K_{2,2}$ or K_4-e ; Stemple [40] has exploited that fact to obtain a structural description and classification of geodetic graphs of diameter two; Kantor [28], working independently and with different terminology, gave the same description (see the discussion following 1.13):

1.7 Proposition. [40] [28] The following statements hold

for every two-connected geodetic graph of diameter two with maximum degree Δ and minimum degree δ in which the largest clique has ω vertices:

- i) Every nontrivial maximal clique is a maximum clique.
- ii) Every vertex has degree Δ or δ .
- iii) If $\delta \neq \Delta$, then $\delta = \Delta - \omega + 2$.
- iv) Every vertex in a maximum clique has degree Δ .
- v) Every vertex at distance 2 from a nontrivial maximal clique has degree Δ .
- vi) G has $1 + \delta\Delta$ vertices.

As can be seen in Figure 3, geodetic graphs of diameter greater than two need not satisfy any of the conditions of the preceding proposition. Particularly in light of the forbidden subgraph characterization, it would seem that the nice behavior of geodetic graphs of diameter two is an anomaly due entirely to the small diameter; we shall return to that question below.

Geodetic graphs of diameter two are not the only ones to have been investigated in detail; Stemple and Watkins [42] have described the geodetic planar graphs, and a result due to Plesník [35] led to a characterization of geodetic graphs homeomorphic to a complete graph (see 4.12-4.14). In view of the apparent intractability of the problem in its original form, Bosak, Kotzig, and Známský [7] considered a variant of Ore's definition:

1.8 Definition. [7] A graph is called strongly geodetic if

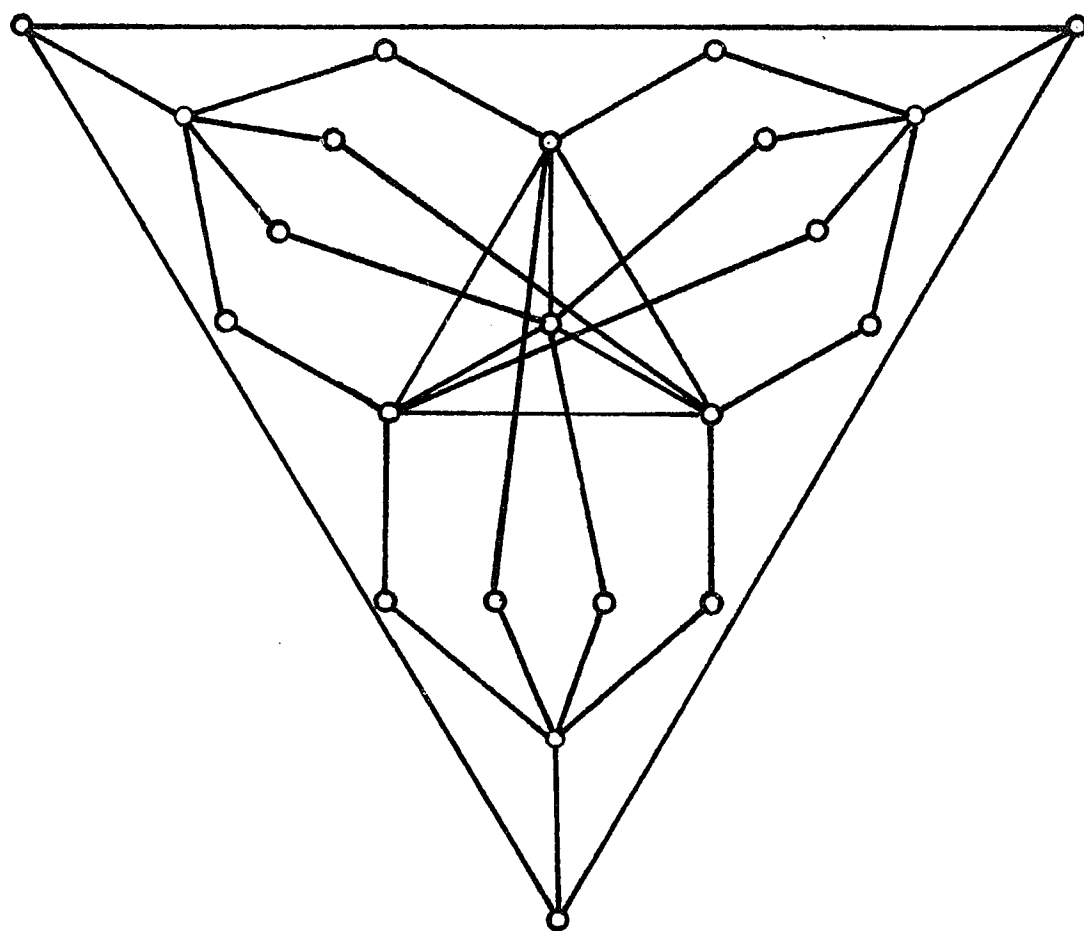


Figure 3.
A geodetic graph of diameter 3.

each two vertices are joined by exactly one path of length not exceeding the diameter of the graph. (Actually, they wrote "at most", rather than "exactly", since they did not require that graphs be connected.) Their stronger condition yields an even nicer structure than that of geodetic graphs of diameter two. In order to state their result, we need the following terminology:

1.9 Definition. [22] A graph with maximal degree Δ is a Moore graph if it satisfies the following equality:

$$(16) \quad |V| = 1 + \Delta \cdot \sum_{i=1}^{\text{diam}} (\Delta-1)^{i-1}.$$

(It is easy to see that every graph satisfies the inequality obtained via replacement of "=" with " \leq " in that definition.)

Singleton [38] proved that all Moore graphs are regular, and characterized them as follows:

1.10 Proposition. [38] A graph is a Moore graph iff $g = 2 \cdot \text{diam} + 1$.

(Every graph with a cycle satisfies the corresponding inequality, $g \leq 2 \cdot \text{diam} + 1$; see 3.2)

Using Singleton's result, Bosak, Kotzig and Znám [7] proved the following:

1.11 Proposition. [7] A (connected) graph is strongly geodetic iff it is a Moore graph or a tree. ■

Moore graphs have been studied extensively, and with great success. Hoffman and Singleton [22] proved that the 7-cycle is the only Moore graph of diameter three, and that there are at most four Moore graphs of diameter two: the 5-cycle, the Petersen graph (see Figure 4), the Hoffman-Singleton graph (see [22]), and possibly one other with 3250 vertices. Independently, Bannai and Ito [2] and Damerell [11] proved that the only Moore graphs of diameter greater than three are the cycles of odd length. As Damerell observed, "...Moore graphs are interesting, and it is a pity there should be so few of them." ([11], p. 227)

In view of the dearth of Moore graphs, it would be nice to find a condition sufficiently strong to preserve much of the beautiful structure of Moore graphs, yet weak enough to admit more graphs. Bose and Dowling [8] introduced a definition couched in the language of finite geometry: a Moore geometry with parameters (a,b) and diameter d is an incidence system of points and lines that satisfies the following three conditions:

- (17) Each point is incident with exactly $(a+1)$ lines.
- (18) Each line is incident with exactly $(b+1)$ points.
- (19) Each two points are joined by a unique chain of length no greater than $2d$.

(Here, a chain is an alternating sequence of distinct points and lines, beginning and ending with a point, such that consecutive pairs are incident; its length is one less than the number of elements)

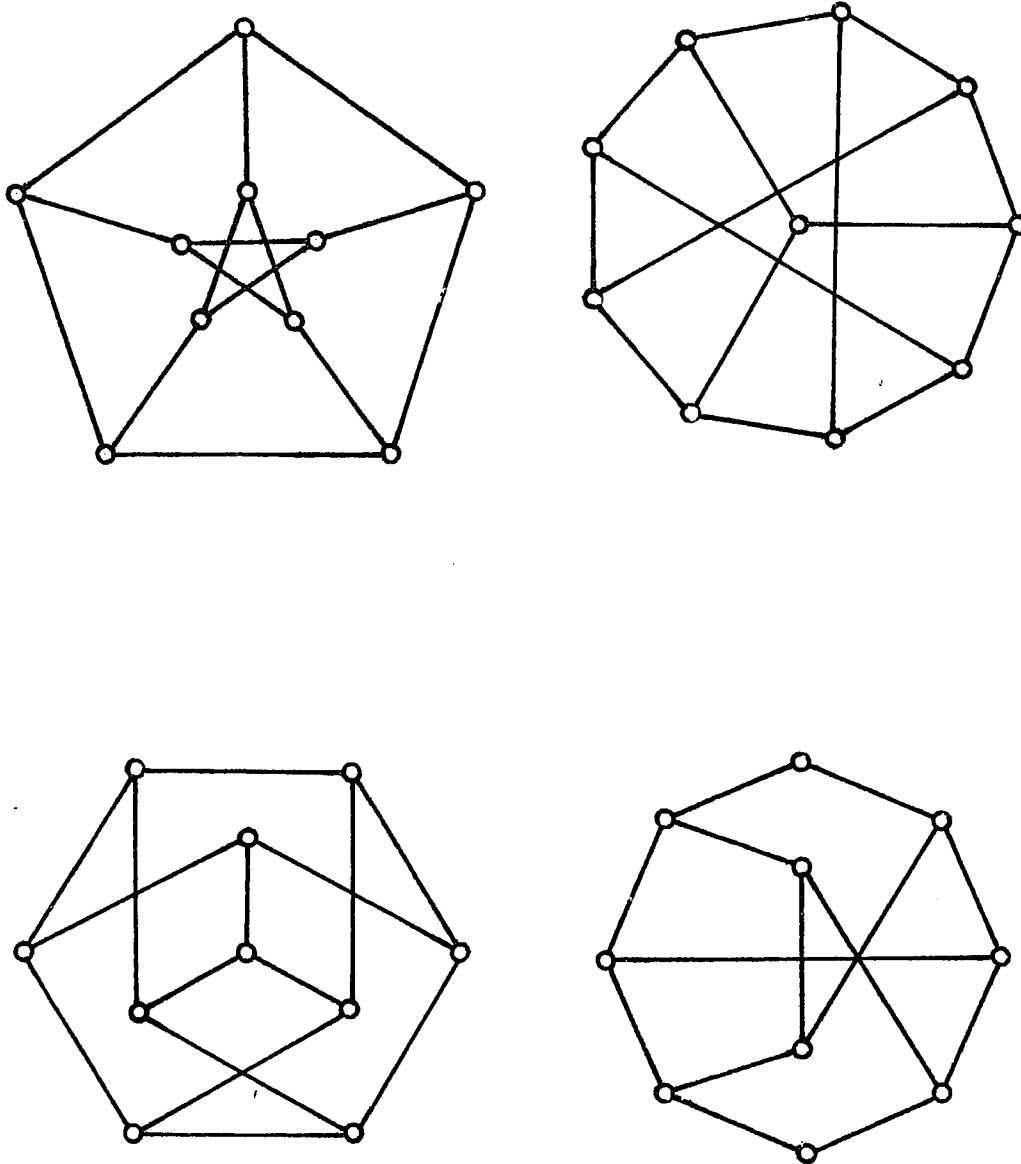


Figure 4.

Four views of the Petersen graph.

Moore geometries of diameter one are just balanced incomplete block designs; furthermore, if $a = b$, then $d = 1$, and the geometry is a finite projective plane (see Chapter VI). Each Moore geometry of diameter greater than 1 may be interpreted as a graph in the following way: the vertices are just the points of the geometry, and two vertices are adjacent if and only if there is a line in the geometry that is incident with both of the corresponding points. In Chapter VI, we show that a graph G corresponds to a Moore geometry if and only if it satisfies the following conditions:

- (20) G is regular of degree $a + 1$.
- (21) Every maximal clique has $b + 1$ vertices.
- (22) Each two vertices are joined by a unique chordless path of length no greater than d .

For $b = 1$, the graphs associated with Moore geometries are precisely the Moore graphs (see [8]). We call a Moore geometry thick if $b > 1$.

Unfortunately, there are no known Moore geometries other than the geometries corresponding to Moore graphs. In fact, Fuglister [15] [16], Damerell and Georgiacois [12], and Damerell [13] have proved the following result, using algebraic methods:

1.12 Proposition. [12] [13] [15] [16] There is no thick Moore geometry of diameter greater than two. ■

Another definition of Moore geometries was offered by

Kantor [28], who defined the points and lines in terms of graphs with these properties: there is more than one vertex; no vertex is adjacent to every other vertex; and two nonadjacent vertices have precisely one common neighbor. Of course, those conditions describe the geodetic graphs of diameter two, and his description of such "Moore geometries" duplicated Stemple's result 1.7. Indeed, that is not an unnatural generalization of Moore graphs of diameter two, for, as Stemple [40] proved, a geodetic graph of diameter two is a Moore graph iff every maximal clique has exactly two vertices. In fact, a geodetic graph of diameter two is a Moore geometry (in the sense of (17) - (19)) iff each maximal clique is a maximum clique (see 6.5).

The disappointing aspect of geodetic graphs of diameter two, both as a well-behaved subclass of the geodetic graphs and as a generalization of Moore graphs, is the artificial restriction to diameter two.

In this work, we introduce a proper subset of the geodetic graphs, the "ultrageodetic" graphs, whose definition is analogous to the definitions of geodetic and strongly geodetic graphs: a graph is ultrageodetic iff each two vertices are joined by a unique chordless path of length no greater than the diameter. This class properly includes the graphs associated with Moore geometries (and thus Moore graphs); for two-connected graphs of diameter one or two, being geodetic is equivalent to being ultrageodetic. In addition to the (ultra-) geodetic graphs of diameter two,

there are nontrivial ultrageodetic graphs of arbitrarily large diameter, and for these we give an explicit construction using finite projective planes.

The glaring similarity of the three definitions of "geodetic" graphs given above suggests a broader underlying concept. In Chapter II, we develop a general notion of geodetic graphs defined in terms of "convexity" in graphs. In addition to its intuitive appeal, the language of convexity leads to a considerable economy of expression.

In Chapter III, we investigate the relationship between shortest cycles in a graph and the uniqueness and extendibility of paths. The results presented there hint at the "extremal" qualities of strongly geodetic and ultrageodetic graphs vis à vis geodetic graphs in general, and provide the technical prerequisites for the proofs of the last three chapters.

The structure of two-connected geodetic graphs is the subject of Chapter IV, which culminates in two "forced subgraph" results; in particular, we show that, in an incomplete geodetic graph G , every complete subgraph K with n vertices is a proper subgraph of a (not necessarily proper) geodetic subgraph H of G that is homeomorphic to K_{n+1} . Although we use the full power of that result only rarely in Chapters V and VI, one corollary proves to be indispensable: incomplete geodetic blocks have no extreme points (4.18; see 2.10 ff.).

The central results are contained in Chapters V and VI,

both of which are concerned with ultrageodetic graphs. In Chapter V, we present an analogue of 1.9 for ultrageodetic graphs (5.2), followed by a collection of technical results in preparation for Chapter VI. Finally, in Chapter VI, we characterize the ultrageodetic graphs of diameter greater than two, and obtain thereby a generalization of 1.7 for ultrageodetic graphs of arbitrary diameter.

Chapter II. Convexity in Graphs

Although the concept of convexity traditionally has been associated with linear spaces, in which betweenness can be defined in terms of linear dependence, a natural notion of betweenness in metric spaces (X, d) was introduced as early as 1928 by Menger [31]: a point $y \in X$ lies between the points $x, z \in X$ if $d(x, y) + d(y, z) = d(x, z)$. In the n -dimensional Euclidean space E^n , a point lies between two others in the sense just defined if and only if it lies on the segment joining them (see [37], p. 3), that is, if and only if it lies between them in the standard sense. A set $S \subseteq X$ is called convex if S contains every point of X that lies between two points of S . (Actually, that is only one of several related definitions of convexity in metric spaces; see [37], pp. 146-148.) Since connected graphs may be viewed as metric spaces, but not as vector spaces, Menger's concept is an obvious candidate for a definition of betweenness and convexity in graphs, and one would expect it to be useful for the study of geometric problems in graph theory.

Convexity in graphs only recently has begun to receive attention in the literature, and the publications that have appeared so far fall roughly into two categories: extremal problems ([1], [3], [4], [18], [19], [20], [21], [32]), and extensions of classical problems and results to graphs

([23], [24], [25], [27]). The questions considered here fall into the former category.

In contrast to previous investigations, our discussion encompasses a variety of related "convexities" rather than just the one defined above. To show how these various types of convexity arise, it is helpful to recast the definition of betweenness in the language of graphs: a vertex y lies between vertices x and z if and only if it lies on an x,z -geodesic. For graphs, then, convex sets are those sets of vertices which are "closed" with respect to the family Γ of geodesics. By replacing Γ with other families Π of paths, we can define various types of convexity. Such an approach has been taken by Jamison [25], who considered the family Φ of chordless paths; the associated notion of convexity is particularly well suited to the study of chordal graphs (see [26]). Since geodetic graphs, strongly geodetic graphs, and trees all can be defined in terms of families of paths, one might expect to find several different "path-convexities" to be of value for the study of geodetic graphs.

Since we shall discuss various types of convexity, we adopt Jamison's terminology [25] for abstract convexity:

2.1 Definition. [25] For an arbitrary set S , an alignment on S is a collection Λ of subsets of S that satisfies the following three conditions:

- i) Both \emptyset and S belong to Λ .
- ii) Λ is closed with respect to intersections.
- iii) Λ is closed with respect to nested unions.

Sets belonging to Λ are called Λ -convex, and the Λ -convex hull of a subset T of S is defined to be

$$(1) \quad \text{hull}_\Lambda(T) := \bigcap \{A \in \Lambda; T \subseteq A\}.$$

For a set A with elements x_1, \dots, x_n , we also write $\text{hull}_\Lambda(x_1, \dots, x_n) := \text{hull}_\Lambda(A)$. For finite sets S , condition iii) is of course superfluous.

Because we are interested in graph-theoretic questions, rather than abstract convexity, we seek to define not just alignments on the vertex-sets of graphs, but ones that are intimately related to the structures of the graphs in question. Our first step toward that goal is to require that the alignments we consider arise from some type of "betweenness", in analogy to Menger's definition; specifically, that "betweenness" is to be defined in terms of a family of paths, as outlined above. Our second step to ensure that the resulting alignment reflects the structure of the graph is to require that the family of paths connect the graph. Finally, we preserve a basic property of geodesics, that every subpath of a geodesic is also a geodesic, by requiring that the family of paths be hereditary.

2.2 Definition. Let Π be an hereditary family of paths in a graph G such that G is Π -connected; such a family is a generating family. The Π -interval between two vertices $x, y \in V$ is the set

$$(2) \quad [x, y]_\Pi := \{v \in V; \forall P(x, y) \in \Pi, v \in P\},$$

and members of $[x, y]_\Pi$ lie Π -between x and y . The

path-alignment on G generated by Π is the collection

$$(3) \quad \Lambda = \Lambda(\Pi) := \{S \subseteq V; \forall x, y \in S, [x, y]_{\Pi} \subseteq S\};$$

Members of Λ are called Λ -convex or Π -convex. A subgraph H of G is called Λ -convex or Π -convex if $V(H)$ is Π -convex.

2.3 Remark. It is a trivial matter to verify that every path-alignment on G is an alignment on V . By definition, a family of paths contains the converse of each of its members, so $[x, y]_{\Pi} = [y, x]_{\Pi}$ for all $x, y \in V$.

2.4 Definition. Let $\Lambda = \Lambda(\Pi)$ be a path-alignment on G . The Λ -convex hull, or Π -convex hull, of a set $A \subseteq V$ is the set

$$(4) \quad \text{hull}_{\Lambda}(A) = \text{hull}_{\Pi}(A) := \bigcap \{C \in \Lambda; A \subseteq C\}.$$

We also use $\text{hull}_{\Pi}(A)$ to denote the subgraph induced by $\text{hull}_{\Pi}(A)$.

2.5 Remark. For an arbitrary set $A \subseteq V$, $\text{hull}_{\Pi}(A)$ is Π -convex, since intersections of Π -convex sets are Π -convex (cf 2.1-ii)); moreover, $A \subseteq \text{hull}_{\Pi}(A)$, so A is Π -convex iff $A = \text{hull}_{\Pi}(A)$. For $x, y \in V$, the fact that $[x, y]_{\Pi}$ is included in every Π -convex set containing x and y implies that $[x, y]_{\Pi} \subseteq \text{hull}_{\Pi}(x, y)$.

In the ensuing discussion of geodetic graphs, we shall refer frequently to the families of paths presented in Table 1. There are graphs for which some or all of those families are identical; for every graph, the inclusions shown in Figure 5 hold, with the proviso that $k \geq \text{diam}$. Those fa-

families are hereditary, and each of them connects G , since Γ does; consequently, each generates a path-alignment on G . For generating families $\Pi_1 \subseteq \Pi_2$, it is easy to see that $\Delta(\Pi_2) \subseteq \Delta(\Pi_1)$; in particular, the hierarchy of Figure 5 gives rise to the dual hierarchy of Figure 6.

$$\begin{aligned}
 \Omega &:= \{P; P \text{ is a path in } G\}. \\
 \Phi &:= \{P \in \Omega; P \text{ is chordless}\}. \\
 \Sigma_k &:= \{P \in \Omega; \lambda(P) \leq k\}, \quad k \geq \text{diam}. \\
 T_k &:= \{P \in \Omega; \lambda(p) \leq k\}, \quad k \geq \text{diam}. \\
 \Sigma &:= \{P \in \Omega; \lambda(P) \leq \text{diam}\} = \Sigma_{\text{diam}}. \\
 T &:= \{P \in \Phi; \lambda(P) \leq \text{diam}\} = T_{\text{diam}}. \\
 \Gamma &:= \{P(x,y) \in \Omega; \lambda(P) = d(x,y)\}.
 \end{aligned}$$

Table 1.

Some generating families for path alignments.

In one sense, the path-alignments considered here must be viewed as being atypical: they are defined for all graphs in an hereditary manner; that is, for each of the families Π of Figure 6, and each $\Pi(G)$ -convex subgraph H of G , every $\Pi(H)$ -convex subgraph is also $\Pi(G)$ -convex. That hereditarily universal nature of the definitions, coupled with the close relationships that they bear to one another (as indicated in Figure 5), permit us to prove results about all of them with a fairly small assortment of techniques.

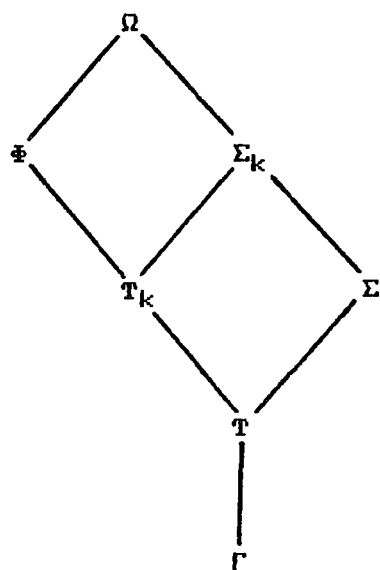


Figure 5.

A hierarchy of generating families.

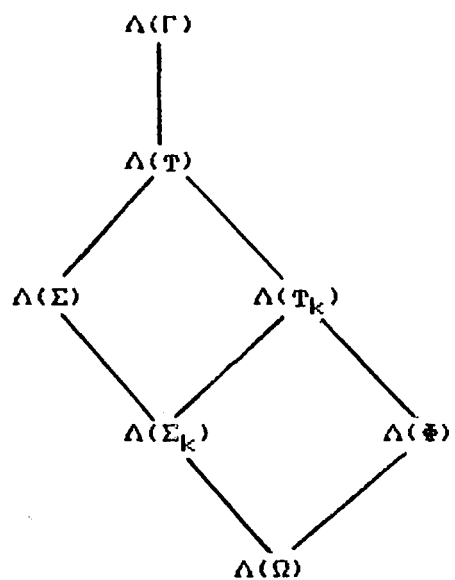


Figure 6.

A hierarchy of path-alignments.

One common property of the families of Table 1 leads to a description of geodetic graphs in terms of convexity: for each generating family Π that includes Γ , every Π -convex subgraph preserves distance in G . It is natural to ask when the converse holds; that is, when is every distance-preserving subgraph a Π -convex subgraph? We begin with a simpler question: when do the vertex-sets of distance-preserving subgraphs form an alignment? Since subgraphs are by definition connected, it is necessary that the intersection of two geodesics be a geodesic; in other words, the graph must be geodetic:

2.6 Proposition. [9] For a graph G , consider the collection $\text{Con}(G)$ of Γ -convex subgraphs, the collection $\text{Met}(G)$ of distance-preserving (metric) subgraphs, and the collection $\text{Geo}(G)$ of geodetic subgraphs. The following statements are equivalent:

- i) G is geodetic.
- ii) $\text{Met}(G) \subseteq \text{Con}(G)$.
- iii) $\text{Met}(G) \subseteq \text{Geo}(G)$.
- iv) $\text{Con}(G) \subseteq \text{Geo}(G)$.

Proof. i) iff ii). If G is geodetic, and if H is a distance-preserving subgraph of G , then H includes the unique geodesic in G joining each two vertices in H , and is therefore Γ -convex. Conversely, if G is not geodetic, then there exist vertices $x, y \in V(G)$ that are joined by at least two different geodesics; each of those geodesics is a distance-preserving subgraph of G that is not Γ -convex.

i) iff iii). If H is a distance-preserving subgraph of G , then a geodesic in H is also a geodesic in G ; hence, if G is geodetic, then H is geodetic as well. On the other hand, if every distance-preserving subgraph of G is geodetic, then G , being a distance-preserving subgraph of itself, is geodetic.

i) iff iv). If H is a convex subgraph of G , then, for each pair of vertices $x, y \in V(H)$, H includes every x, y -geodesic in G . Thus H is a distance-preserving subgraph of G ; by the preceding argument, H must be geodetic if G is. If every convex subgraph of G is geodetic, then G , being a convex subgraph of itself, is geodetic. ■

A similar result holds in the general case:

2.7 Proposition. For a generating family of paths Π that includes Γ , the following statements are equivalent:

- i) Every distance-preserving subgraph is Π -convex.
- ii) G is geodetic and $\Pi = \Gamma$.
- iii) Each pair of vertices $x, y \in V$ determines a unique path $P(x, y) \in \Pi$.
- iv) Every Π -convex hull of two vertices induces a path.
- v) Every path in Π induces a Π -convex subgraph.

Proof. i) \Rightarrow ii). Fix vertices $x, y \in V$ and a path $P(x, y) \in \Gamma \subseteq \Pi$. By i), the geodesic P , being a distance-preserving subgraph, is Π -convex. As a result, $P(x, y) \in \Gamma$ is the only x, y -path in Π for each pair $x, y \in V$. It follows that the graph is geodetic and $\Pi = \Gamma$.

ii) \Rightarrow iii). This is just the definition of a geodetic graph.

iii) \Rightarrow iv). Fix vertices $x, y \in V$, and consider the unique path $P(x, y) \in \Pi$. Since $\Gamma \subseteq \Pi$, there must be an x, y -geodesic in Π ; but P is the only x, y -path in Π , so it must be a geodesic. In particular, P is an induced (i. e., chordless) x, y -path. According to iii), each subpath $P|_{v, w}$ is the unique v, w -path in Π ; therefore, $[v, w]_{\Pi} \subseteq V(P)$ for all $v, w \in V(P)$, so that $V(P)$ is Π -convex. But $\text{hull}_{\Pi}(x, y)$ is the intersection of all convex sets containing x and y , so $\text{hull}_{\Pi}(x, y) \subseteq V(P)$. Furthermore, since $P \in \Pi$, we know that $V(P) \subseteq [x, y]_{\Pi}$. By Remark 2.5, $[x, y]_{\Pi} \subseteq \text{hull}_{\Pi}(x, y)$, and we conclude that $V(P) = \text{hull}_{\Pi}(x, y)$.

iv) \Rightarrow v). Fix a path $P(x, y) \in \Pi$. According to 2.2 and 2.5, $V(P) \subseteq [x, y]_{\Pi} \subseteq \text{hull}_{\Pi}(x, y)$. By iv), $\text{hull}_{\Pi}(x, y)$ induces a path, say Q ; since Q is convex, $P = Q|_{x, y}$. The convexity of Q implies further that $[v, w]_{\Pi} \subseteq Q$ for all $v, w \in V(Q)$, so the fact that each subpath $Q|_{v, w}$ is the only v, w -path in Q means that $Q|_{v, w}$ must be the only v, w -path in Π . Thus, for all $v, w \in V(Q)$, $[v, w]_{\Pi} \subseteq V(Q|_{v, w})$. As a result, for all $s, t \in V(Q|_{v, w})$, $[s, t]_{\Pi} \subseteq V(Q|_{s, t}) \subseteq V(Q|_{v, w})$; in other words, $Q|_{v, w}$ is convex, and $V(Q|_{v, w}) = \text{hull}_{\Pi}(v, w)$. Since

$$(5) \quad P = Q|_{x, y} = \text{hull}_{\Pi}(x, y) = Q,$$

we see that P induces a convex subgraph.

v) \Rightarrow i). Fix vertices x and y in a distance-pre-

serving subgraph H . Then H includes a geodesic $P(x,y)$; by hypothesis, $P \in \Gamma \subseteq \Pi$. According to v), P is convex; consequently, $[x,y]_\Pi \subseteq V(P) \subseteq V(H)$, and it follows that H is a convex subgraph. ■

Although the preceding two results do not provide a characterization of graphs in which every distance-preserving subgraph is Π -convex, they do suggest a name for such graphs:

2.8 Definition. For a path-alignment $\Delta(\Pi)$ on G , the graph G is Π -geodetic if each pair of vertices $x,y \in V(G)$ determines a unique path $P(x,y) \in \Pi$.

2.9 Remark. For generating families $\Pi_1 \subseteq \Pi_2$, a Π_2 -geodetic graph is also Π_1 -geodetic. Thus the inclusions of Figure 5 yield the implications of Figure 7. The Σ -geodetic graphs are just the strongly geodetic graphs (see 1.8); we shall call the T -geodetic graphs ultrageodetic.

As we investigate the structure of Γ -, Σ_k -, and T_k -geodetic graphs in the next three chapters, we shall frequently make use of the concept of an extreme point, which is a point in a convex set S such that $S \setminus \{x\}$ is convex.

2.10 Remark. Since the intersection of two convex sets is convex, convex subsets of convex sets inherit extreme points: if S and T are convex sets such that $S \subseteq T$, and if $x \in S$ is an extreme point of T , then the set $S \setminus \{x\} = S \cap T \setminus \{x\}$ is convex, and it follows that x is an extreme point of S . For a path-alignment $\Delta(\Pi)$ on a graph, a vertex v is an extreme point if and only if

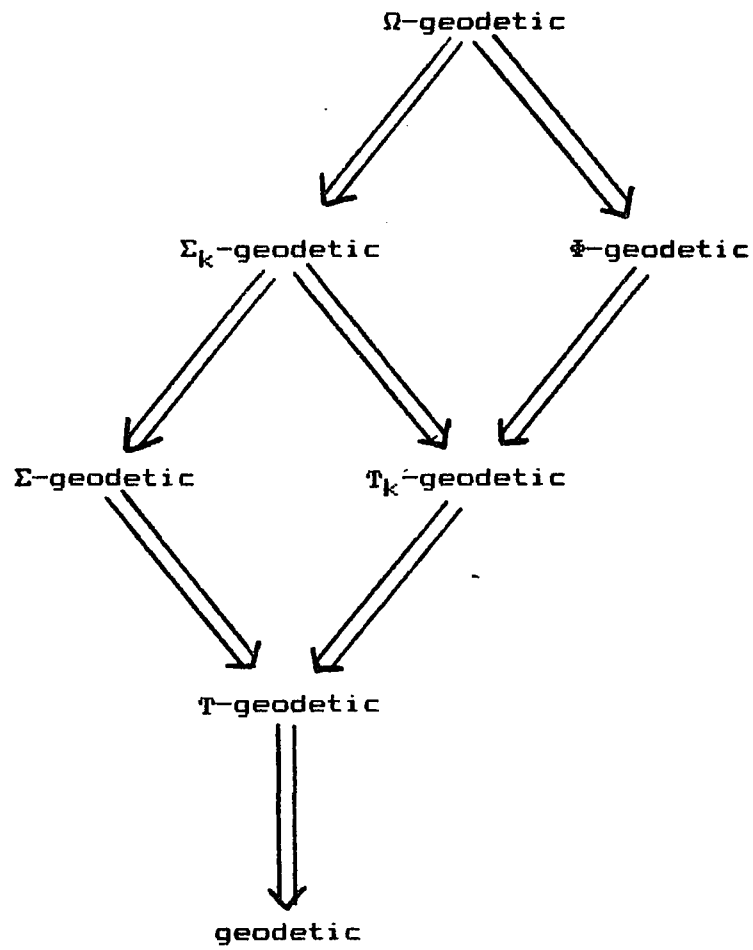


Figure 7.

A hierarchy of implications for graphs.

$[x, y]_{\Pi} \subseteq V \setminus \{v\}$ for all vertices $x, y \in V \setminus \{v\}$. By definition, the latter condition holds for precisely those vertices v such that $v \notin [x, y]_{\Pi}$ for all $x, y \neq v$. In other words, a vertex is an extreme point if and only if it does not lie Π -between two other vertices. In particular, for the alignments generated by Ω , Σ , and Σ_k , an extreme point is simply a vertex that is not the internal vertex of some path of length two, i. e., an endpoint. However, for other alignments, there may be other endpoints:

2.11 Lemma. For each generating family Π such that $\Gamma \subseteq \Pi \subseteq \Phi$, the following statements are equivalent:

- i) The vertex x is an extreme point.
- ii) The punctured neighborhood $S(v; 1)$ is a clique.
- iii) The vertex x is not an internal vertex of a geodesic; i. e., x does not lie Γ -between other vertices.
- iv) There is a unique maximal clique $M \subseteq V$ that contains x , namely, the neighborhood $M = B(x; 1)$.

Proof. The equivalence of ii), iii), and iv) is obvious.

i) iff iv). If a vertex x lies in only one maximal clique, then x is not an internal vertex of a chordless path; consequently, since $\Pi \subseteq \Phi$, x cannot lie Π -between two other vertices. By the Remark preceding the Lemma, x is an extreme point. Conversely, if x lies in two different maximal cliques K and L , then there exist nonadjacent vertices $w \in K$ and $y \in L$; the geodesic $w - x - y$ belongs to Π , since $\Gamma \subseteq \Pi$; again by the Remark preceding

the Lemma, x is not an extreme point. ■

From this point on, we shall refer to extreme points only with respect to path-alignments $\Lambda(\Pi)$ for which $\Gamma \subseteq \Pi \subseteq \Phi$, so that 2.11 is applicable.

When an extreme point is removed from a tree or from K_4-e , the resulting convex subgraph may have an extreme point that was not previously an extreme point. However, we show that that cannot happen if the graph is two-connected and has no subgraph isomorphic to K_4-e :

2.12 Lemma. Let G be a two-connected graph with no subgraph isomorphic to K_4-e , and let H be the subgraph of G obtained via the removal of all extreme points of G . Then H is a convex subgraph with no extreme points.

Proof. By 2.10, each extreme point of G that lies in a convex subgraph is also an extreme point of that subgraph. Thus, if we remove the extreme points of G one at a time (in any order), each of the sequence of subgraphs so obtained will be convex. In particular, H is convex. Now suppose that H has an extreme point, say v . By 1.5, the fact that G has no K_4-e implies that the punctured neighborhood of v in G is a disjoint union of cliques. Because v is an extreme point of H , exactly one of those, say K , meets $V(H)$; all of the others are included in $V(G) \setminus V(H)$. Since v is not an extreme point of G , there is at least one clique of the latter sort, say L . But each vertex in L is an extreme point of G , and thus has no neighbor outside L other than v . It follows that v is

a cut-vertex, in contradiction to the assumption that G is two-connected. ■

Since we are interested in the uniqueness of x,y -paths in Π , we frequently discuss distinct paths $P(x,y)$ and $Q(x,y)$; it is useful to know that such paths may be assumed to be internally disjoint:

2.13 Lemma. If there exist distinct paths $P(x,y)$ and $Q(x,y)$, then there are distinct internally disjoint subpaths $P|v,w$ and $Q|v,w$.

Proof. Let $P(x,y)$ and $Q(x,y)$ be distinct paths, and consider the last vertex v on P such that $P|x,v = Q|x,v$. It may be that $v = x$; however, $v \neq y$, since P and Q are distinct. Now let w be the first vertex after v on P that also lies on Q ; perhaps $w = y$. By the choice of v , $P|v,w$ and $Q|v,w$ are distinct; by the choice of w , they are internally disjoint. ■

2.14 Corollary. If an hereditary family of paths Π contains two x,y -paths, then it contains two internally disjoint v,w -paths for some $v,w \in V$. ■

According to 1.3, trees are precisely the graphs in which each two vertices are joined by a unique path. Thus, we have a characterization of Ω -geodetic graphs:

2.15 Proposition. A graph is Ω -geodetic iff it is a tree. ■

Similarly, the characterization of block graphs as graphs in which each block is complete (1.4) permits a sim-

ple description of Φ -geodetic graphs:

2.16 Proposition. A graph is Φ -geodetic iff it is a block graph.

Proof. Suppose that G is not Φ -geodetic, and let $P(x,y)$ and $Q(x,y)$ be chordless paths. By 2.13, we may assume that P and Q are internally disjoint. Then PQ' is a cycle and lies entirely within an incomplete block of G . Now suppose that G is not a block graph. Then there exist nonadjacent vertices x and y that lie in the same block. Since the block is not an edge, it must be two-connected, and it follows that x and y lie on a cycle. Among the cycles containing x and y is at least one of minimum length, say C ; since x and y are not adjacent, $\lambda(C) > 3$. The choice of C with minimum length ensures that the two paths joining x and y in C are chordless, and we conclude that G is not Φ -geodetic. ■

Chapter III. Uniqueness and Extendibility of Paths

Throughout this investigation, we deal with two properties that a path in a family Π may or may not enjoy: uniqueness and extendibility. That is, for a particular $P(x,y) \in \Pi$, we would like to be able to answer the following questions:

- i) Is P the only x,y -path in Π ? (If so, then P is Π -unique.)
- ii) Do there exist vertices w and z such that both wP and Pz belong to Π ? (If so, then P is Π -extendible.)

With suitable restrictions on the path under consideration, those questions can be answered via reference to the minimum length of a certain kind of cycle, and it is the purpose of this chapter to provide the requisite technical observations and results.

The minimum length of a cycle in a graph is the girth g ; if the graph has no cycles, then $g := \infty$. Since a chord in a cycle yields two new cycles, each of length less than that of the original cycle, every cycle of length g must be chordless; in fact, it must be distance-preserving as well:

3.1 Lemma. Every cycle of length $g(G)$ preserves distance in G .

Proof. Suppose that C_1 is a (chordless) cycle of length g that does not preserve distance in G , and let $P(x,y)$ be a geodesic in C_1 that is not a geodesic in G . Then there is a path $Q(x,y) \neq P$ in G with $\lambda(Q) < \lambda(P) \leq \lfloor \lambda(C_1)/2 \rfloor$. By 2.13, P and Q have internally disjoint subpaths $R := P|_{v,w}$ and $S := Q|_{v,w}$; it follows that the cycle $C_2 := R(S')$ has length

$$\begin{aligned}
 (1) \quad \lambda(C_2) &= \lambda(R) + \lambda(S) \\
 &\leq \lambda(P) + \lambda(Q) \\
 &< 2\lambda(P) \\
 &\leq g,
 \end{aligned}$$

in contradiction to the definition of g . ■

3.2 Corollary. If G has a cycle, then $g \leq 2 \cdot \text{diam} + 1$.

Proof. Let C be a cycle of length g ; by 3.1, we have that $\text{diam}(C) \leq \text{diam}(G)$, and it follows that

$$(2) \quad g = \lambda(C) \leq 2 \cdot \text{diam}(C) + 1 \leq 2 \cdot \text{diam}(G) + 1. \quad \blacksquare$$

The girth of a graph provides a means for determining whether a path $P \in \Sigma_k$ is Σ_k -unique or Σ_k -extendible:

3.3 Lemma. The following statements hold for an arbitrary path $P(x,y)$ in a graph G :

- i) If $\lambda(P) < \lfloor g/2 \rfloor$, then every other x,y -path in G is longer than P .
- ii) If $\lambda(P) < g - 1$, and if G has no endpoints, then there exist vertices w and z in G such that wP and Pz are paths.

Proof. i) Suppose that i) does not hold in G ; then there

exist paths $P(x,y)$ and $Q(x,y)$ such that

$$(3) \quad \lambda(Q) \leq \lambda(P) < \lceil g/2 \rceil.$$

By 2.13, they have internally disjoint subpaths $R := P|_{v,w}$ and $S := Q|_{v,w}$. For the cycle $C := R(S')$, we have that

$$\begin{aligned} (4) \quad \lambda(C) &= \lambda(R) + \lambda(S) \\ &\leq \lambda(P) + \lambda(Q) \\ &\leq (\lceil g/2 \rceil - 1) + (\lceil g/2 \rceil - 1) \\ &< g, \end{aligned}$$

in contradiction to the definition of g .

ii) Let $P(x,y)$ be a path of length less than $g - 1$, and suppose that G has no endpoints. Now y is adjacent to exactly one vertex of P , for otherwise it would lie in a cycle of length less than $g - 1$. Since y is not an endpoint, it must have a neighbor z lying outside P , and Pz is the desired path. The existence of w follows similarly. ■

3.4 Proposition. The following statements hold for every path P in Σ_k :

- i) If $k < \lceil g/2 \rceil$, then P is Σ_k -unique.
- ii) If $\lambda(P) < \min\{g-1, k\}$, and if G has no endpoints, then P is Σ_k -extendible.

Proof. i) Suppose that $k < \lceil g/2 \rceil$, and that $P(x,y)$ and $Q(x,y)$ are distinct members of Σ_k ; without loss of generality, we may assume that $\lambda(P) \leq \lambda(Q)$. Now Q belongs to Σ_k , so $\lambda(Q) \leq k$, and it follows from part i) of Lemma 3.1 that every other x,y -path must be longer than Q ; in parti-

cular, P must be longer than Q , in contradiction to the assumption.

ii) Under the stated conditions, there must be vertices w and z such that wP and Pz are paths, by part ii) of Lemma 3.1. Both of those paths belong to Σ_k , since $\lambda(P) < k$. ■

With the help of 3.4, we can characterize graphs that are geodetic with respect to the k -strong and strong alignments:

3.5 Lemma. A graph is Σ_k -geodetic iff $g \geq 2k + 1$. ■

Proof. Suppose first that G is Σ_k -geodetic, and let x and y be opposite vertices in a cycle C of length g . Then there are two x, y -paths in C of length not exceeding $(g+1)/2$, so it must be that $(g+1)/2 \geq k+1$; it follows that $g \geq 2k+1$. If G has no cycle, then $g = \infty > 2k+1$.

Suppose now that $g \geq 2k+1$. Then $\lfloor g/2 \rfloor \geq k+1$, and 3.4-i) yields the fact that every Σ_k -path is Σ_k -unique, as desired. ■

For $k = \text{diam}$, we obtain as a corollary the following result of Bosak, Kotzig, and Známl [7]:

3.6 Corollary. [7] A graph is strongly geodetic iff $g \geq 2 \cdot \text{diam} + 1$. ■

3.7 Proposition. The Σ_k -geodetic graphs are precisely the trees of diameter not exceeding k and the strongly geodetic graphs of diameter k .

Proof. In order for the alignment $\Lambda(\Sigma_k)$ to be defined, G

must be Σ_k -connected; that is the case iff $k \geq \text{diam}$. By 3.5 and 3.2, a graph with a cycle is Σ_k -geodetic iff it satisfies the inequality

$$(5) \quad 2k + 1 \leq g \leq 2 \cdot \text{diam} + 1.$$

Since $k \geq \text{diam}$, that inequality holds iff $g = 2 \cdot \text{diam} + 1$ and $\text{diam} = k$, and the assertion follows by 3.6. ■

3.8 Lemma. In an Σ_k -geodetic graph with no endpoints, every path of length less than k is Σ_k -extendible.

Proof. Let $P(x,y)$ be a path of length less than k in an Σ_k -geodetic graph with no endpoints. By 3.5,

$$(6) \quad \lambda(P) < k \leq (g-1)/2,$$

and 3.4-ii) tells us that P is Σ_k -extendible. ■

For $k = \text{diam}$, the preceding Lemma yields this:

3.9 Corollary. In a strongly geodetic graph, every path of length less than diam is Σ -extendible. ■

For graphs with girth 3, the results 3.1–3.8 describe properties of cycles of length 3 and paths of length 1, and thus are of no value whatsoever for the investigation of such graphs; in the following analogues of those results, the additional assumption that the paths considered are chordless provides a means for "ignoring" the cycles of length 3. For example, the graph in Figure 8 has four cycles of length 3; however, all of the locally chordless cycles have length at least 9, and every chordless path of length less than 4 can be extended to a chordless path of length 4. In order to state results for chordless paths,

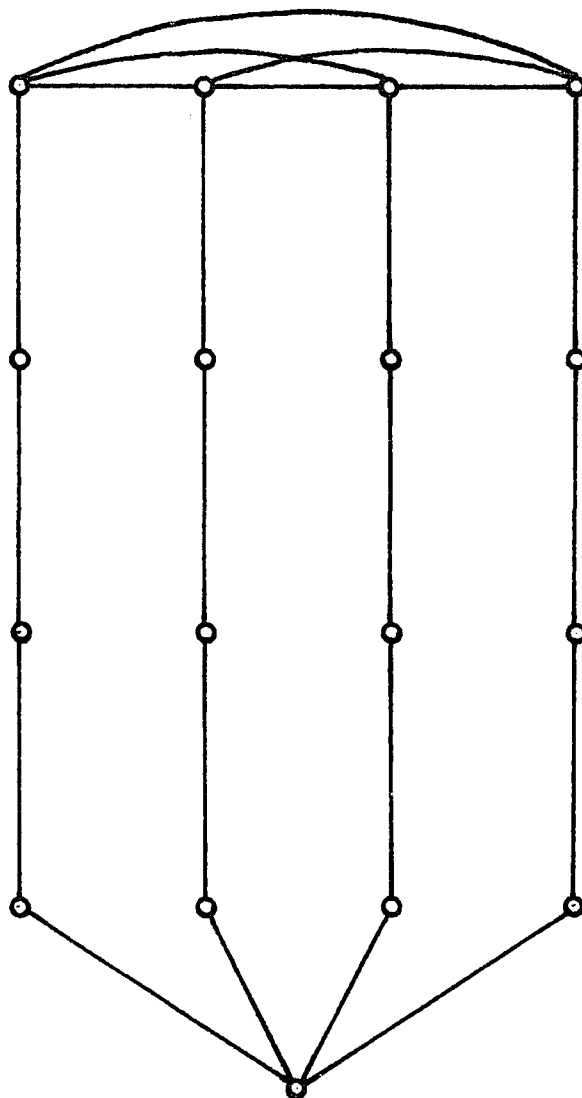


Figure 8.
A geodetic graph with $g = 3$,
 $g^* = 9$, and $\text{diam} = 4$.

we need a new concept of "girth" that "ignores" cycles of length 3:

3.10 Definition. [9] The augmented girth g^* of a graph is the minimum length among locally chordless cycles; if no such cycle exists, then $g^* := g$.

Since the definition specifically excludes triangles if any other locally chordless cycles exist, the augmented girth is a measure of shortest cycles in which distinct edges lie in distinct maximal cliques. Maximal cliques must be taken into account in other ways in the discussion of T_k -convexity. For example, in the analogues of 3.1 - 3.8 for paths in T_k , we must introduce the additional condition on G that no edge lies in two maximal cliques; i.e., the graph $K_4 - e$ is a forbidden subgraph. However, the assumption that G has no extreme points (e.g., in 3.17) is not new: for the alignments Σ_k of the preceding results, the extreme points are simply endpoints (cf 2.10). The restriction to chordless paths and cycles also complicates the proof of 3.17, which is the analogue of 3.3 for paths in T_k ; although the argument is by no means difficult, we shall apply it sufficiently frequently to warrant formalizing it at the outset. We begin with a weak version.

3.11 Lemma. Every locally chordless cycle of length g^* is chordless.

Proof. Consider a locally chordless cycle

$$(7) \quad C: x_1 - x_2 - \dots - x_n - x_1$$

that has a chord, and set

$$(8) \quad m := \min\{i; \{x_i, x_j\} \in E(G) \text{ for some } j < i - 1\}$$

and

$$(9) \quad k := \max\{i < m - 1; \{x_i, x_m\} \in E(G)\}.$$

Then $(k, m) \neq (1, n)$, since C has a chord, and $k < m - 2$, since C is locally chordless. Therefore,

$$(10) \quad x_k - x_{k+1} - \dots - x_m - x_k$$

is a locally chordless cycle whose length is strictly less than that of C ; consequently, $\lambda(C) > g^*$. ■

3.12 Definition. A double chordless path, written DCP, is a graph D whose distinct vertices can be given the labels $x, y, v_1, \dots, v_m, w_1, \dots, w_n$ ($n \geq m \geq 1$) in such a way that

$$(11) \quad P_1: x - v_1 - \dots - v_m - y$$

and

$$(12) \quad P_2: y - w_1 - \dots - w_n - x$$

are chordless paths in D . We shall say that P_1 and P_2 determine D . The length of D is

$$(13) \quad \lambda(D) := |V(D)| = m + n + 2;$$

if we wish to emphasize the lengths of two particular determining paths, then we shall refer to D as an m, n -DCP.

The cycle

$$(14) \quad x - v_1 - \dots - v_m - y - w_1 - \dots - w_n - x$$

is called a canonical cycle of D . By a DCP in a graph, we shall mean an induced subgraph that is a DCP.

A DCP may have several different pairs of determining

paths, and several different canonical cycles (see Figure 9). Now we are ready for the aforementioned generalization of 3.11.

3.13 Proposition. For each graph such that $4 < g^* < \infty$, the following statements are equivalent:

- i) g^* is the minimum length of a DCP.
- ii) For each DCP D of minimum length, every canonical cycle of D is locally chordless.
- iii) G has no subgraph isomorphic to $K_4 - e$.

Moreover, if $g^* = 4$, then statement i) is true, and ii) and iii) are equivalent.

Proof. Suppose that $4 < g^* < \infty$.

i) \Rightarrow iii). The graph $K_4 - e$ is a DCP of length 4.

iii) \Rightarrow ii). Suppose that $P: x = v_1 - \dots - v_m = y$ and $Q: y = w_1 - \dots - w_n = x$, with $n \geq m \geq 1$, determine a DCP D of minimum length in G , and consider the canonical cycle $C = PQ$. If $n = 1$, then C must be locally chordless, for G has no subgraph isomorphic to $K_4 - e$. For $n > 1$, if C were not locally chordless, then either $\{v_1, w_n\}$ or $\{w_1, v_m\}$ would have to be an edge of G . If $\{v_1, w_n\}$ were an edge, then we could construct a DCP of length strictly less than $\lambda(D)$: either $m = 1$, so that the paths $w_n = v_1 = y$ and $Q(y, w_n)$ would determine a DCP of length $\lambda(D) - 1$, or $m > 1$, in which case, for

$$(15) \quad k := \min\{i; \{v_1, w_i\} \in E\},$$

the paths $P(v_1, y)$ and $(Q(y, w_k) \cup v_1)$ would determine a DCP

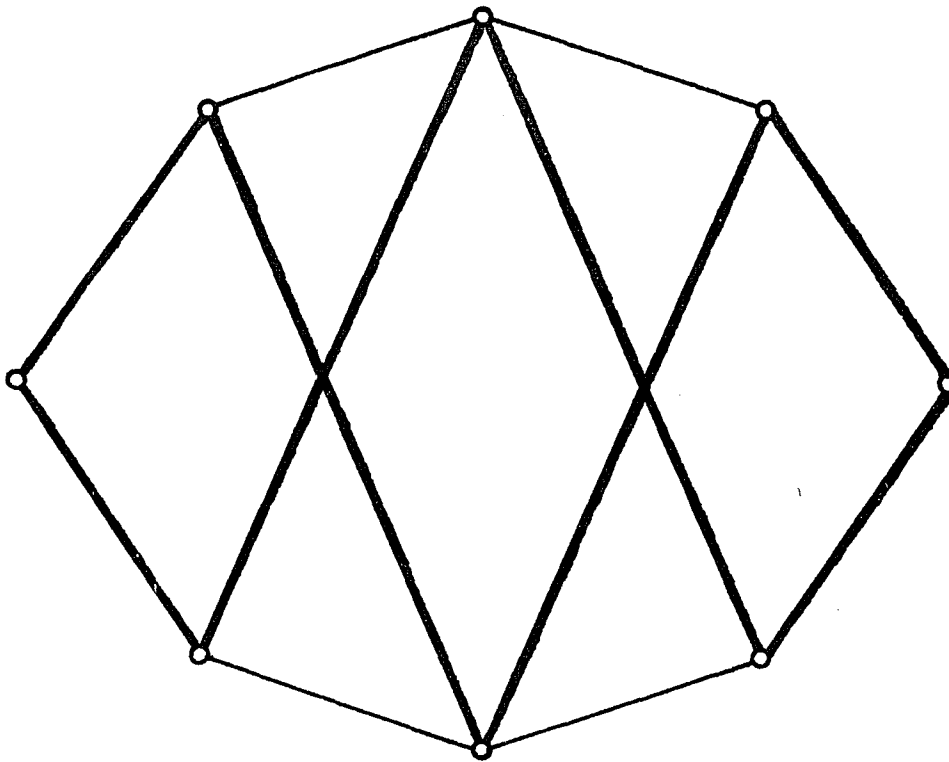


Figure 9.

two canonical cycles in a DCP. The edges around the perimeter form one canonical cycle, and the edges indicated by the heavy lines form another.

of length $m + k + 1 < m + n + 2 = \lambda(D)$. That would contradict our choice of D with minimum length, so it must be that $\{v_1, w_n\} \notin E$. A similar argument demonstrates that $\{w_1, v_m\} \notin E$, and we conclude that C is locally chordless.

ii) \Rightarrow i). On the one hand, ii) implies that g^* is no greater than the length of a shortest DCP in G ; on the other hand, we know by 3.11 that each locally chordless cycle of length g^* is a chordless cycle, and therefore a DCP. Statement i) follows immediately.

Now suppose that $g^* = 4$. Then there is a chordless cycle C of length 4, and C is a DCP of length 4. Since every DCP has length at least 4 (by definition), i) must hold; the equivalence of ii) and iii) follows easily from the observation that $K_4 - e$ is a DCP. ■

The augmented girth also provides a new means for the description of block graphs:

3.14 Proposition. A graph is a block graph iff g^* is 3 or ∞ and there is no subgraph isomorphic to $K_4 - e$.

Proof. Suppose first that G is a block graph. Since each block is complete (1.4), every chordless cycle must have length 3, and G has no subgraph isomorphic to $K_4 - e$. Now suppose that G is not a block graph; then G is not a tree, so $g^* \neq \infty$. Let K be a maximal clique in an incomplete block B of G . Then there exist adjacent vertices $x \in K$ and $y \in V(B) \setminus K$; since B is two-connected, x is not a cut-vertex. Thus, there must be a path $P(y, v)$ joining y and K such that $v \neq x$ is the only vertex in K

that lies on P . Now the path xP has its endpoints in K and all internal vertices in $V(B) \setminus K$; among the paths with that property, let $Q(s,t)$ be one of minimum length. The minimality of $\lambda(Q)$ ensures that $\{s,t\}$ is the only chord of Q . If G has no subgraph isomorphic to $K_4 - e$, then K is the unique maximal clique that includes the edge $\{s,t\}$, by 1.5. It follows that the length of the chordless cycle Qs is greater than 3, and thus also that $g^* > 3$. ■

3.15 Lemma. [9] If G has no subgraph isomorphic to $K_4 - e$, then every locally chordless or chordless cycle of length g^* is a distance-preserving subgraph.

Proof. We observe first that the statement makes sense for locally chordless cycles, since those of length g^* are in fact induced subgraphs, by 3.11. The assertion is trivial for $g^* \leq 5$, and vacuous for $g^* = \infty$. Suppose that $5 < g^* < \infty$, and let C be a (locally) chordless cycle of length $g^*(G)$ that does not preserve distances in G . We shall prove that G has a subgraph isomorphic to $K_4 - e$. Choose vertices $x, y \in V(C)$ such that $d_C(x, y) > d_G(x, y)$, and let $P(x, y)$ and $Q(x, y)$ be geodesics in G and C , respectively. By 2.13, there are internally disjoint subpaths $R := P|_{v,w}$ and $S := Q|_{v,w}$; since R and S are chordless, they determine a DCP D . We see that $\lambda(D) < \lambda(C) = g^*$, so it follows from 3.13 that G has a subgraph that is isomorphic to $K_4 - e$. ■

3.16 Corollary. If G has a cycle, then $g^* \leq 2 \cdot \text{diam} + 1$.

Proof. Suppose that G has a cycle, and let C be a cycle

of length g^* . By 3.15, C is a distance-preserving subgraph of G . Hence, $\text{diam}(G) \geq \text{diam}(C) \geq \lfloor g^*/2 \rfloor \geq (g^*-1)/2$, and we conclude that $g^* \leq 2 \cdot \text{diam}(G) + 1$. ■

3.17 Lemma. The following statements hold for an arbitrary chordless path $P(x,y)$ in a graph G with no subgraph isomorphic to K_4-e :

- i) If $\lambda(P) < \lfloor g^*/2 \rfloor$, then every other x,y -path is longer than P .
- ii) If $\lambda(P) < g^* - 2$, and if G has no extreme points, then there exist vertices w and z such that wP and Pz are chordless paths.

Proof. i) Suppose that the assertion is false; then there is a chordless path $Q(x,y)$ such that

$$(16) \quad \lambda(Q) \leq \lambda(P) < \lfloor g^*/2 \rfloor \leq (g^*+1)/2.$$

By 2.13, there are internally disjoint subpaths $R := P|_{v,w}$ and $S := Q|_{v,w}$; we need only observe that R and S' determine a DCP D of length

$$(17) \quad \lambda(D) = \lambda(R) + \lambda(S) \leq \lambda(P) + \lambda(Q) < g^*,$$

in contradiction to 3.13.

ii). Suppose that G has no extreme points, and that $P(x,y)$ is a chordless path of length less than $g^* - 2$. Let v be the predecessor of y in P ; the fact that P is chordless ensures that v is the only neighbor of y in P . By 1.5, the fact that G has no subgraph isomorphic to K_4-e implies that there is a unique maximal clique K containing both v and y . According to 2.11, since y is

not an extreme point, it must be adjacent to some vertex z lying outside K ; by 1.5, z has at most one neighbor in K , and therefore is not adjacent to v . If z were adjacent to some other vertex in P , then, for the closest such vertex u to y in P , the paths Plu, y and $y - z - u$ would determine a DCP of length less than g^* , in contradiction to Proposition 3.13. ■

3.18. Proposition. Let $P(x, y)$ be a T_k -path in a graph G with no subgraph isomorphic to $K_4 - e$. Then the following statements hold:

- i) If $k < \lfloor g^*/2 \rfloor$, then P is T_k -unique.
- ii) If $\lambda(P) < \min\{g^*-2, k-1\}$, and if G has no extreme points, then P is T_k -extendible.

Proof. i) Suppose that $k < \lfloor g^*/2 \rfloor$. If the assertion is not true, then there is a path $Q(x, y) \neq P$ in T_k ; since the length of each is less than $\lfloor g^*/2 \rfloor$, 3.17-i) implies the nonsensical statement that the length of each exceeds that of the other.

ii). According to 3.17-ii), the present hypotheses are sufficient to ensure the existence of chordless paths wP and Pz . Since $\lambda(P) < k$, each of those paths belongs to T_k . ■

3.19 Lemma. A graph G with $g^* > 3$ is T_k -geodetic iff $g^* \geq 2k + 1$ and G has no subgraph isomorphic to $K_4 - e$.

Proof. Suppose first that G is T_k -geodetic. Then G is Γ -geodetic, by 2.9, and therefore has no subgraph isomorphic to $K_4 - e$ (see Figure 2). Let x and y be opposite ver-

tices in a cycle C of length $g^* > 3$. In C , there are two chordless paths of length not exceeding $g^*/2$ joining x and y . Since G is T_k -geodetic, it must be that $(g^*+1)/2 \geq k+1$; in other words, $g^* \geq 2k+1$. If G has no cycle of length g^* , then $g^* = \infty > 2k+1$.

Now suppose that G has no subgraph isomorphic to K_4-e , and that $g^* \geq 2k+1$. Then $\lceil g^*/2 \rceil \geq g^*/2 \geq k+1/2$, so that $k < \lceil g^*/2 \rceil$. It follows from 3.18-i) that every path in T_k is T_k -unique. ■

For $k = \text{diam}$, 3.19 yields this:

3.20 Corollary. In order for a graph to be ultrageodetic, it is necessary and sufficient that it satisfy one of the following two conditions:

- i) G has no K_4-e and $g^* \geq 2 \cdot \text{diam} + 1$.
- ii) G is a block graph.

Proof. Consider first a graph G with $g^* = 3$. On the one hand, if G is ultrageodetic, then it is geodetic (see Figure 7), so it has no K_4-e (see Figure 2); it follows from 3.14 that G is a block graph. On the other hand, if i) holds, then $\text{diam} = 1$, so G is complete, and ii) holds, by 1.4; if ii) holds, then G is \mathbb{Z} -geodetic, by 2.16, and therefore ultrageodetic (see Figure 7).

Now suppose that $g^* > 3$. If ii) holds, then G has no cycles; that is, $g^* = \infty$, and i) holds. By 3.19, i) holds iff G is ultrageodetic. ■

3.21 Proposition. The T_k -geodetic graphs are precisely the block graphs of diameter not exceeding k and the ultrageo-

detic graphs of diameter k .

Proof. The alignment $\Lambda(T_k)$ is defined iff G is T_k -connected; that is the case iff $\text{diam} \leq k$. By 3.19 and 3.16, a graph with $g^* > 3$ is T_k -geodetic iff it has no subgraph isomorphic to $K_4 - e$ and it satisfies the inequality

$$(18) \quad 2k + 1 \leq g^* \leq 2 \cdot \text{diam} + 1.$$

That inequality holds iff $g^* = 2 \cdot \text{diam} + 1$ and $\text{diam} = k$; the assertion follows by 3.20. ■

3.22 Lemma. In a T_k -geodetic graph with no extreme points, every chordless path of length less than k is extendible.

Proof. Let $P(x,y)$ be a chordless path of length less than k in a T_k -geodetic graph with no extreme points. By 3.19, $k \leq (g^* - 1)/2$, so $k \leq g^* - 2$; since $\lambda(P) < k$, part ii) of Proposition 3.18 is applicable: P is T_k -extendible. ■

For $k = \text{diam}$, 3.22 reduces to this:

3.23 Corollary. In an ultrageodetic graph with no extreme points, every chordless path of length less than diam is T -extendible. ■

Chapter IV. Metric Structure in Geodetic Graphs

Although the $(\Gamma-)$ geodetic graphs of Ore's original problem have been studied extensively, most of the known results concerning their structure deal with special cases, e. g., planar ones [42], ones of diameter two [40], and homeomorphs of the complete graph K_n [41]. Of those which pertain to all geodetic graphs, most are little more than simple observations (one noteworthy exception is Stemple's theorem that every suspended path in a two-connected geodetic graph is a geodesic; see 4.14). Of the following observations, 4.1 and 4.3 belong to the folklore of the subject, having been (re-)discovered by virtually everyone who has considered the problem; 4.3 and 4.4 are essentially restatements of the definition. We begin by recalling what we saw in Figure 2:

4.1 Lemma. A geodetic graph has no subgraphs isomorphic to K_4-e or $K_{2,2}$.

Proof. One need only observe that each of these graphs has two paths of length two joining the same nonadjacent vertices (see Figure 2). ■

4.2 Lemma. [9] If G is geodetic, then g^* is odd or infinite.

Proof. Suppose that g^* is finite, and let C be a locally chordless cycle of length g^* . By 4.1, since G is geodetic, it has no subgraph isomorphic to K_4-e ; hence, ac-

cording to 3.15, C is a distance-preserving subgraph of G . For opposite vertices on C , there will be a unique geodesic joining them in C , and thus also in G , iff the length of C is odd. ■

4.3 Lemma. [39] A graph G is geodetic iff, for each pair of vertices $x, y \in V(G)$ and each nonnegative integer k not exceeding $d(x, y)$, there is a unique vertex z such that $d(x, z) = k$ and $d(y, z) = d(x, y) - k$.

Proof. Suppose first that G is not geodetic; then there exist distinct geodesics $P(x, y)$ and $Q(x, y)$, which we may assume to be internally disjoint, by 2.14. Let v and w be the first vertices after x on P and Q , respectively. Then $d(v, y) = d(w, y) = d(x, y) - 1$, $d(x, v) = 1$, and $d(x, w) = 1$. Now suppose that, for some $x, y \in V$, there are distinct vertices v and w such that $d(x, v) = d(x, w) = k$ and $d(v, y) = d(w, y) = d(x, y) - k$. Consider geodesics $P(x, v)$, $Q(v, y)$, $R(x, w)$, and $S(w, y)$. Since PQ and RS each have at most $1 + d(x, y)$ vertices, they are x, y -geodesics; but $PQ \neq RS$, so G is not geodetic. ■

4.4 Lemma. A graph G is geodetic iff each two distinct vertices x and y determine a unique maximal clique M containing x such that $d(M, y) = d(x, y) - 1$.

Proof. Suppose first that there exist vertices x and y such that x lies in two maximal cliques L and M with $d(L, y) = d(M, y) = d(x, y) - 1$. If G has a subgraph that is isomorphic to $K_4 - e$, then it is not geodetic, by 4.1; suppose, then, that G has no such subgraph. According to

1.5, it follows that L and M have no vertex other than x in common. Since each contains a vertex that is closer to y than x is, there are distinct vertices $v \in L$ and $w \in M$ such that $d(v,y) = d(w,y) = d(x,y) - 1$. Hence, by 4.3, G is not geodetic.

Suppose now that G is not geodetic. If G has a subgraph isomorphic to K_4-e , then the two vertices of degree three in that subgraph have the desired property. Hence, we assume that G has no subgraph isomorphic to K_4-e . Let (x,y) be a pair of vertices such that x has two neighbors that are closer to y than x is, and, with respect to that condition, such that $d(x,y)$ is minimal. Consider two x,y -geodesics

$$(1) \quad x - v_1 - \dots - v_m - y$$

and

$$(2) \quad x - w_1 - \dots - w_m - y;$$

due to the minimality of $d(x,y)$, they are internally disjoint. If $\{v_1, w_1\} \notin E$, then the maximal cliques containing x and v_1 , and x and w_1 , respectively, are both closer to y than x is; similarly, if $\{v_m, w_m\} \notin E$, then the maximal cliques containing y and v_m , and y and w_m , respectively, are both closer to x than y is. In each of those cases, the result follows. If both of those pairs belong to E , then we get a contradiction: either $m = 1$, in which case x, y, v_1 , and w_1 induce a K_4-e , or $m > 1$, in which case there are two paths of length m joining w_1 and v_m , namely,

$$(3) \quad w_1 - v_1 - v_2 - \dots - v_m$$

and

$$(4) \quad w_1 - w_2 - \dots - w_m - v_m;$$

hence, either $d(w_1, v_m) = m$, so that the pair (w_1, v_m) provides a contradiction to the choice of (x, y) , or $d(w_1, v_m) < m$, in which case w_1 and v_1 are distinct neighbors of x that are closer to v_m than x is, and the pair (x, v_m) provides a contradiction to the choice of (x, y) . ■

Despite their simplicity, 4.3 and 4.4 are quite useful; in particular, they lead to a considerable economy of expression via the following terminology:

4.5 Definition. For vertices x and y in a geodetic graph G , and a nonnegative integer $k \leq d(x, y)$, we define $\langle x, y; k \rangle = \langle x, y; k \rangle_G$ to be the (unique, by 4.3) vertex in $S(x; k) \cap B(y; d(x, y) - k)$. For distinct vertices x and y in a geodetic graph, $M(x, y)$ is defined to be the (unique, by 4.4) maximal clique containing x and $\langle x, y; 1 \rangle$, and $m(x, y) := |M(x, y)|$.

4.6 Remark. In the preceding definition, we actually have that

$$(5) \quad S(x; k) \cap B(y; d(x, y) - k) = S(x; k) \cap S(y; d(x, y) - k);$$

since that condition is symmetric in x and y , we see that $\langle x, y; k \rangle = \langle y, x; d(x, y) - k \rangle$. For $n \geq 0$, if $d(x, y) = n + 1$, then $\langle x, y; n \rangle \in M(y, x) \subseteq \{\langle x, y; n \rangle\} \cup R(x, \langle x, y; n \rangle; n + 1)$, and $m(y, x) \leq 1 + r(x, \langle x, y; n \rangle; n + 1)$. Furthermore, $v = \langle x, y; n \rangle$

iff $d(x,v) = n$ and x, v , and y satisfy the triangle equality

$$(6) \quad d(x,y) = d(x,v) + d(v,y).$$

4.7 Lemma. In a geodetic graph, $y = \langle z, x; j \rangle$ for some j iff, for every $k \leq d(x,y)$, $\langle x, z; k \rangle = \langle x, y; k \rangle$.

Proof. Suppose first that $\langle x, z; k \rangle = \langle x, y; k \rangle$ for every $k \leq d(x,y)$. In particular, then,

$$(7) \quad \langle x, z; d(x,y) \rangle = \langle x, y; d(x,y) \rangle = y,$$

and the assertion follows immediately.

Now we suppose that $y = \langle z, x; j \rangle$, and fix $0 \leq k \leq d(x,y)$. Then two applications of the triangle equality of 4.6 and two applications of the triangle inequality show that

$$\begin{aligned} (8) \quad d(z,x) &= d(z,y) + d(y,x) \\ &= d(z,y) + d(y, \langle x, y; k \rangle) + d(\langle x, y; k \rangle, x) \\ &\geq d(z, \langle x, y; k \rangle) + d(\langle x, y; k \rangle, x) \\ &\geq d(z,x). \end{aligned}$$

Since the first and last terms are equal, equality holds throughout. That is, $z, \langle x, y; k \rangle$, and x satisfy the triangle equality

$$(9) \quad d(z,x) = d(z, \langle x, y; k \rangle) + d(\langle x, y; k \rangle, x);$$

that and the fact that $d(\langle x, y; k \rangle, x) = k$ imply that

$$(10) \quad \langle x, y; k \rangle = \langle x, z; k \rangle. \quad \blacksquare$$

The structure of geodetic graphs may be described in terms of the reaches of vertices and maximal cliques; for our discussion of ultrageodetic graphs, that turns out to be

an indispensable fact. We begin with yet another virtual restatement of the definition:

4.8 Lemma. A graph is geodetic iff $R(x,y) \cap R(x,z) = \emptyset$ for each vertex x and each pair of distinct neighbors y and z of x .

Proof. Suppose that w lies in $R(x,y) \cap R(x,z)$. Then w satisfies the following triangle equalities, by 1.1:

$$(11) \quad d(x,w) = d(x,y) + d(y,w).$$

$$(12) \quad d(x,w) = d(x,z) + d(z,w).$$

As x is adjacent to y and z , we see that $y = \langle x,w;1 \rangle$ and $z = \langle x,w;1 \rangle$, so y and z are not distinct neighbors of x . ■

The next three lemmata provide the technical foundation for the fundamental result 4.17, which is used extensively in the succeeding chapters.

4.9 Lemma. [9] Let K be a maximal clique in a geodetic graph. If a vertex z lies in both $R(K,x)$ and $R(K,y)$ for distinct vertices $x,y \in K$, then z lies in $\bigcap \{R(K,w;d(K,z)); w \in K\} \subseteq \bigcap \{R(K,w); w \in K\}$.

Proof. Suppose that such vertices x, y , and z exist, and set $n := d(K,z)$; since $z \in R(K,x)$, it follows from 1.2 that it satisfies the triangle equality

$$(13) \quad n = d(K,z) = d(K,x) + d(x,z) = d(x,z),$$

and, as $z \in R(K,y)$,

$$(14) \quad n = d(K,z) = d(K,y) + d(y,z) = d(y,z).$$

Fix a vertex w in K different from x and y . Since

$d(K,z) = n$, it must be that $n \leq d(w,z) \leq n+1$; moreover, w has two neighbors at distance n from z , namely, x and y . By 4.3, it follows that $d(w,z) = n$. ■

4.10 Corollary. For distinct vertices x and y lying in a maximal clique K of a geodetic graph, and a natural number n , $R(K,x;n) \cap R(K,y;n) = \bigcap \{R(K,w;n); w \in K\}$.

Proof. As the inclusion " \supseteq " is obvious, we need only verify the inclusion " \subseteq ". Fix $z \in R(K,x;n) \cap R(K,y;n)$ for some n ; then $d(K,z) = n$, and it follows from 4.9 that z lies in $\bigcap \{R(K,w;n); w \in K\}$. ■

4.11 Lemma. [9] Fix a natural number n and a maximal clique K in a geodetic graph, and suppose that, for $k = 1, \dots, n$, $\bigcap \{R(K,x;k); x \in K\} = \emptyset$. Then, for each pair of distinct vertices $y, z \in K$, there is no edge joining a vertex in $R(K,y;n)$ with a vertex in $R(K,z;n)$.

Proof. We observe first that $R(K,y;n) \cap R(K,z;n) = \emptyset$, by 4.10. Consider vertices $v \in R(K,y;n)$ and $w \in R(K,z;n)$. Since $w \notin R(K,y;k)$ for $k \leq n$, it must be that $d(y,w) = n+1$. Consequently, $z = \langle y, w; 1 \rangle$, and 4.7 implies that $\langle w, y; 1 \rangle = \langle w, z; 1 \rangle$. But $\langle w, y; 1 \rangle \in S(K; n-1)$; in particular, $\langle w, z; 1 \rangle = \langle w, y; 1 \rangle \neq v$. By 4.3, $\langle w, y; 1 \rangle$ is the unique neighbor of w whose distance from y does not exceed $d(y,w) - 1 = n$. Specifically, since $d(v,y) = n$, it must be that v and w are not adjacent. ■

4.12 Lemma. [9] Let K be a maximal clique in a two-connected geodetic graph G without extreme points. Then there exists a positive integer n such that the set

$\cap \{R(K, x; n); x \in K\}$ is nonempty.

Proof. Suppose that no such n exists, and fix a vertex $y \in K$. For each $x \neq y$ in K , 4.10 implies that

$$(15) \quad R(K, x) \cap R(K, y) = \emptyset;$$

furthermore, 4.11 and 1.5 imply that, for each $x \neq y$ in K , y is the only vertex in $R(K, y)$ that is adjacent to some vertex in $R(K, x)$. Thus y is the only vertex in $R(K, y)$ that has a neighbor in $V \setminus R(K, y)$. Since y is not an extreme point, it is not the only member of $R(K, y)$; consequently, y must be a cut-vertex of G , in contradiction to the assumption that G is two-connected. ■

We turn now to a special family of geodetic graphs, the "pyramids" (the terminology is due to Stemple [40]). For a function f from $V(K_n)$ into the nonnegative integers, let $G(n, f)$ be the homeomorph of K_n obtained by subdividing each edge $\{x, y\}$ of K_n with $f(x) + f(y)$ new vertices.

In addition to defining the graphs $G(n, f)$, Plesník [35] proved the following result:

4.13 Proposition. [35] Every $G(n, f)$ is geodetic. Furthermore, if $n = 3$ or $n = 4$, then every geodetic graph that is homeomorphic to K_n is isomorphic to $G(n, f)$ for some f . ■

Zelinka [44] observed that 4.13 holds for all values of n if an additional condition is imposed:

4.14 Proposition. [44] For arbitrary n , a geodetic homeomorph of K_n in which every suspended path is a geodesic

must be isomorphic to $G(n, f)$ for some f . ■

That "extra" condition turns out to hold for all geodesic graphs, as was proved by Stemple in [41] (a weaker version had been proved by Stemple and Watkins [42]):

4.15 Proposition. [41] In a two-connected geodetic graph, every suspended path is a geodesic. ■

4.16 Remark. If the support of f consists of precisely one vertex $a \in V(K_{n+1})$, then the graph $P = G(n+1, f)$ is called a (regular) pyramid with base $K = K_n$ and apex a (see Figures 8 and 10; Stemple's definition [40] is formulated differently). If $f(a) = k$, then $\text{diam}(P) = k + 1$; moreover, for distinct vertices $x, y \in K$, $R(K, x)$ is just the set of vertices of the suspended path joining x and a , and $R(K, x) \cap R(K, y) = \{a\}$. Due to the fact that $g^*(P) = 2 \cdot \text{diam}(P) + 1$, pyramids are ultrageodetic. Finally, we note that pyramids have no extreme points, since each of their vertices lies in at least two maximal cliques (see 2.11). We are interested in pyramidal subgraphs, i. e., induced subgraphs that are isomorphic to pyramids.

4.17 Lemma. [9] Suppose that G is a geodetic graph, and let P be a pyramidal subgraph of G with base K , apex a , and diameter k . Then P preserves distance in G iff $d_G(K, a) = k$.

Proof. The direction "only if" is obvious. Suppose that $d_G(K, a) = k$; we wish to show that $d_P(v, w) = d_G(v, w)$ for each pair of vertices $v, w \in V_P$. Since $g^*(P) = 2k + 1$, the absence of extreme points in P ensures that every geo-

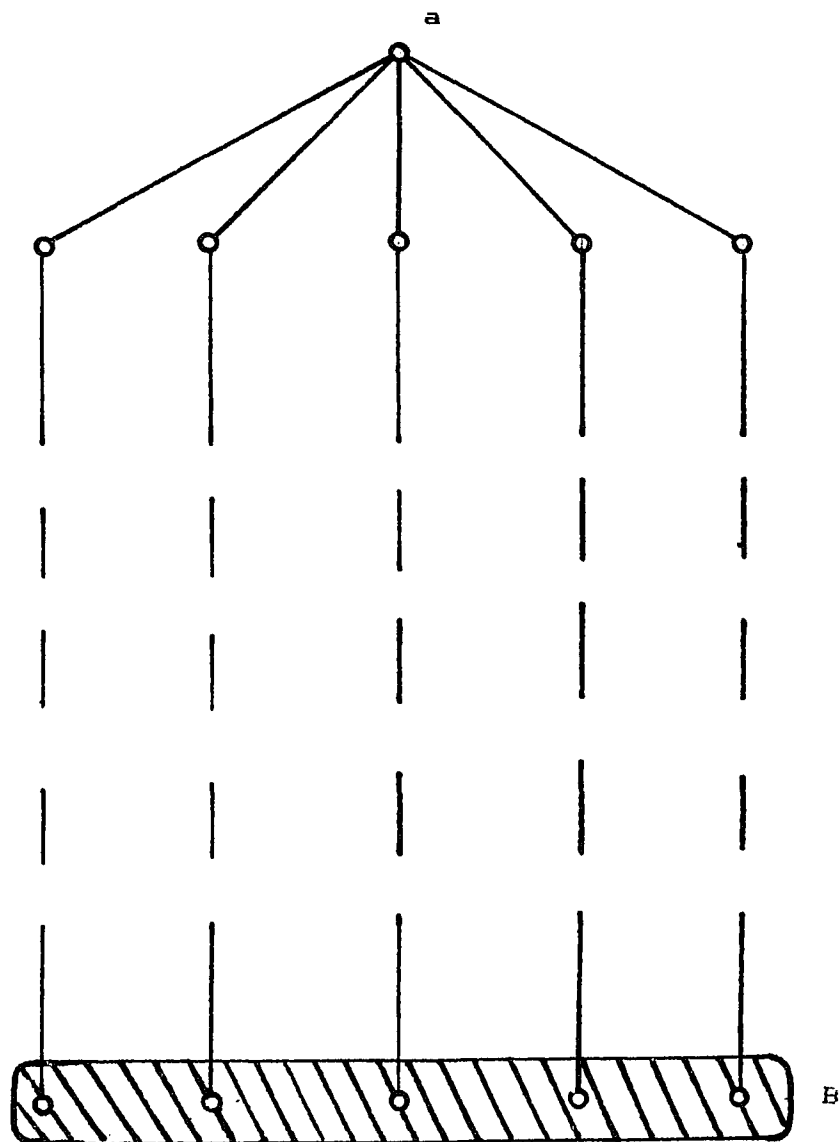


Figure 10.

Construction of a pyramid with base $B = K_n$, apex a , and diameter $k > 1$. In this case, $n = 5$, and the dashed lines indicate the $k - 2$ remaining internal vertices of each suspended path joining a with the vertices of B .

desic of length less than k in P is extendible, by 3.19 and 3.22; hence, it suffices to show that $d_G(v,w) \geq k$ for all vertices v and w such that $d_P(v,w) = k$. For each $x \in K$, label the vertices of the associated suspended path in P as follows:

$$(16) \quad x = x_0 - x_1 - \dots - x_{k-1} - x_k = a.$$

Now each vertex in P is an x_n for some $x \in K$ and some n ; moreover, for distinct vertices $x, y \in K$,

$$(17) \quad d_P(x_i, y_j) = \min\{i+j+1, 2k-i-j\}.$$

It follows that, for fixed x and $n < k$,

$$(18) \quad d_P(x_n, y_j) = k \text{ iff } j \in \{k-n, k-n-1\}.$$

Fix distinct vertices $x, y \in K$; we shall prove by induction on n that the following two statements hold for each n such that $0 \leq n \leq k-1$:

$$(19) \quad d_G(x_n, y_{k-n}) \geq k.$$

$$(20) \quad d_G(x_n, y_{k-n-1}) \geq k.$$

(Of course, each inequality implies the corresponding equality, since $\text{diam}(P) = k$.) For the case in which $n = 0$, we have that $d_G(x_0, y_k) = d_G(x, a) = k$ by assumption. Since $x_{k-1} = \langle x_0, y_k; k-1 \rangle$, Lemma 4.3 implies that $d_G(x_0, y_{k-1}) \geq k$. Now (19) and (20) hold for $n = 0$; fix $m \geq 1$, and suppose that they hold for $n = m-1$. Then $d_G(x_{m-1}, y_{k-m}) = k$, so $x_{m-2} = \langle y_{k-m}, x_{m-1}; k-1 \rangle$, unless $m = 1$, in which case $y = \langle y_{k-m}, x_{m-1}; k-1 \rangle$; in both cases, by 4.3, $d_G(x_m, y_{k-m}) \geq k$. But then $y_{k-m+1} = \langle x_m, y_{k-m}; k-1 \rangle$, and another application of 4.3 yields that $d_G(x_m, y_{k-m+1}) \geq k$. Thus conditions (19)

and (20) hold for each n and arbitrary distinct vertices $x, y \in K$. ■

We come now to our main result on the structure of a $(\Gamma-)$ geodetic block: either it is complete, or each clique is the base of a distance-preserving pyramid. Consequently, each vertex lies in the base of a pyramid and in the maximal clique determined by the first edge on its suspended path. Hence, by 2.11, incomplete geodetic blocks have no extreme points; that simple fact is crucial for verifying the extendibility of paths (cf Chapter 3) and for the application of earlier results in this section.

4.18 Proposition. [9] In an incomplete two-connected geodetic graph, each maximal clique K is the base of a distance-preserving pyramidal subgraph.

Proof. Let X be the set of extreme points of an incomplete two-connected geodetic graph G , and consider the subgraph H of G induced by $V(G) \setminus X$; since extreme points of G do not lie between other points of G (2.11), H preserves distances in G . Since G is incomplete, H is non-trivial; by 2.12, the fact that G has no subgraph isomorphic to K_4-e implies that H has no extreme points.

Choose a maximal clique K in H , and set

$$(21) \quad n := \min \{m; \cap \{R(K, x; m); x \in K\} \neq \emptyset\}.$$

(The existence of n is ensured by 4.12.) Fix a vertex a in $\cap \{R(K, x; n); x \in K\}$, and consider the geodesics

$$(22) \quad Q_x: x = \langle x, a; 0 \rangle - \langle x, a; 1 \rangle - \dots - \langle x, a; n \rangle = a$$

for $x \in K$. Since $\langle x, a; k \rangle \in R(K, x)$ and $\langle y, a; m \rangle \in R(K, y)$, then, if they are equal, $k \geq n$, by 4.9. It follows that the paths Q_x are internally disjoint. Therefore, the vertices $\langle x, a; m \rangle$, with $x \in K$ and $m \leq n$, induce a pyramidal subgraph P of H with base K , apex a , and diameter n ; by Lemma 4.17, the fact that the paths Q_x are geodesics implies that P preserves distances in H . We have seen that each maximal clique K in H is the base of a distance-preserving pyramidal subgraph of H , so the result holds if $G = H$. Suppose, on the contrary, that there is some extreme point $p \in X$, and consider the clique $K = S(p; 1) \setminus X$; since G is two-connected, K contains at least two vertices. Since G is geodetic, it has no subgraph isomorphic to $K_4 - e$, so K is a maximal clique of H ; the preceding construction yields a distance-preserving subgraph P of H with base K . Let a and k be the apex and diameter of P , respectively. Since no extreme point lies between two other vertices (by 2.11), each vertex of K lies on a shortest p, a -path; but K has at least two vertices, in contradiction to 4.3. We conclude that $X = \emptyset$, and hence that $H = G$. ■

4.19 Corollary. [9] An incomplete geodetic block has no extreme points. ■

Despite their rather technical appearance, the final three results of this chapter have a useful intuitive interpretation based on the idea of "suspending" a graph: We picture each of the vertices as having the same "mass", and

each of the edges as having the same "length"; for an arbitrary vertex x or maximal clique K , we can "suspend" the graph from x or K to obtain the configurations of Figure 11. In the next chapter, we shall refer frequently to such figures in order to give intuitive meaning to technical results; however, the figures play no role in the proofs. In the next result, we find that, if we suspend an incomplete geodetic block from a vertex y , then the outermost sphere is composed of bases of pyramids "pointed at" y (see Figure 12).

4.20 Proposition. Let y be a vertex of an incomplete geodetic block. Then the following conditions are satisfied:

- i) For each $x \in S(y, e(y))$, there is a maximal clique K containing x for which $d(K, y) = e(y)$.
- ii) Each maximal clique K such that $d(K, y) = e(y)$ is the base of a distance-preserving pyramid with apex a such that $d(y, K) = d(y, a) + d(a, K)$.

Proof. i) Fix $x \in S(y, e(y))$; since x is not an extreme point (4.19), it must lie in at least two maximal cliques, by 2.11. By 4.4, for each maximal clique $K \neq M(x, y)$ containing x , $d(K, y) = d(x, y) = e(y)$.

ii) Let K be a maximal clique such that $d(K, y) = e(y)$. For each $k \geq 0$, set

$$(23) \quad A_k := S(y; e(y) - k) \cap (\cap \{R(K, x; k); x \in K\};$$

observe that $y \in A_{e(y)}$. Define $n := \min\{k; A_k \neq \emptyset\}$, and choose $a \in A_n$. Then $d(y, K) = d(y, a) + d(a, M)$. (It may be that $a = y$.) The minimality of n ensures that the a, K -

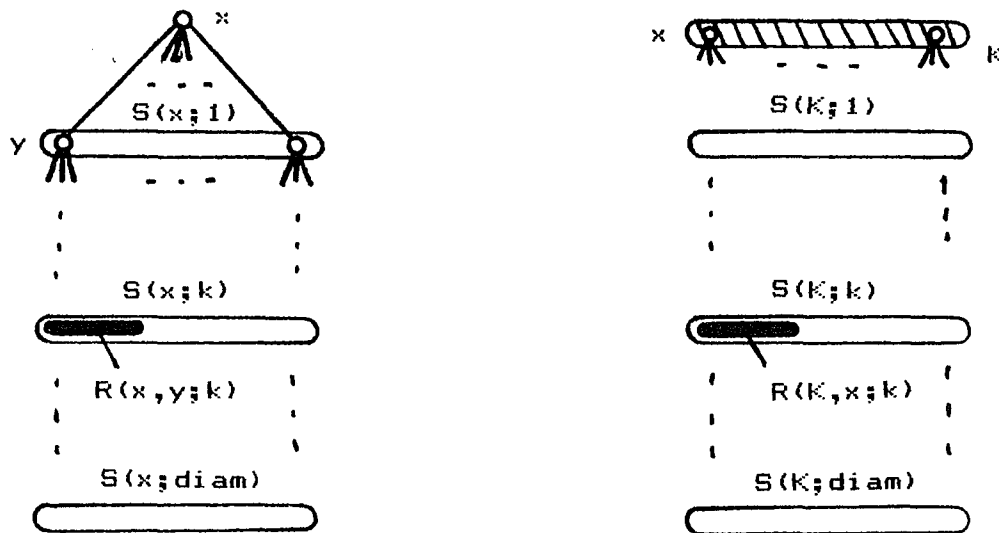


Figure 11.

Suspension of a graph from a
vertex or a maximal clique.

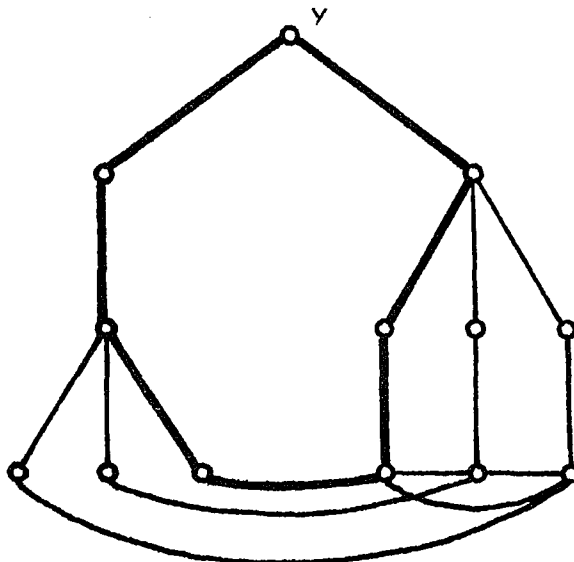


Figure 12.

A geodetic graph suspended from a vertex y . The heavy lines indicate a pyramid "pointed" at y .

geodesics are pairwise internally disjoint. It follows from 4.11 that a , M , and the paths between them determine a pyramidal subgraph P of G . Finally, by 4.17, P preserves distances in G . ■

The next two results confirm that, if we suspend a geodetic graph from a maximal clique K , then the reaches of K and the reaches of its vertices are situated as we expect them to be.

4.21 Lemma. Fix a vertex x in a maximal clique K of a geodetic graph and a vertex $v \in R(K, x; k)$. Then, for each nonnegative integer $j \leq k$, $d(K, \langle x, v; j \rangle) = j$.

Proof. Using only the definitions and the triangle inequality, we deduce that

$$\begin{aligned}
 (24) \quad d(K, v) &= d(x, v) \\
 &= d(x, \langle x, v; j \rangle) + d(\langle x, v; j \rangle, v) \\
 &\geq d(K, \langle x, v; j \rangle) + d(\langle x, v; j \rangle, v) \\
 &\geq d(K, v).
 \end{aligned}$$

Since the first and last terms are identical, equality must hold throughout, and we conclude that

$$(25) \quad d(K, \langle x, v; j \rangle) = d(x, \langle x, v; j \rangle) = j. \quad \blacksquare$$

4.22 Proposition. Suppose that K is a maximal clique in a geodetic graph. For each vertex $x \in K$, a positive integer k , and each vertex $v \in R(K, x; k)$, $R(K, v; k+1) = R(x, v; k+1)$.

Proof. Suppose first that $s \in R(K, v; k+1)$. From the definitions, Remark 1.2, and the triangle inequality, it follows that

$$\begin{aligned}
 (26) \quad d(x,s) &\geq d(K,s) \\
 &= d(K,v) + d(v,s) \\
 &= d(x,v) + d(v,s) \\
 &\geq d(x,s).
 \end{aligned}$$

Since the first and last terms are identical, equality holds throughout. Thus s satisfies the appropriate triangle equality (1.1), and therefore lies in $R(x,v)$; since $d(v,s) = 1$ (by assumption), it follows that $s \in R(x,v;k+1)$.

Now suppose that $s \in R(x,v;k+1)$. According to 4.7, then, $\langle x,v;1 \rangle = \langle x,s;1 \rangle$. By 4.21, $d(\langle x,v;1 \rangle, K) = 1$; in particular, $\langle x,v;1 \rangle \notin K$. For each vertex $w \neq x$ in K , w is not adjacent to $\langle x,v;1 \rangle$, by 1.5; but $\langle x,v;1 \rangle = \langle x,s;1 \rangle$ is the unique neighbor of x whose distance from s does not exceed $d(x,s) - 1$ (by 4.3), so $d(w,s) \geq d(x,s)$. It follows that $d(K,s) = d(x,s)$, and we infer from the definitions that

$$\begin{aligned}
 (27) \quad d(K,s) &= d(x,s) \\
 &= d(x,v) + d(v,s) \\
 &= d(K,v) + d(v,s) \\
 &= k + 1;
 \end{aligned}$$

as a consequence, $s \in R(K,v;k+1)$. ■

Chapter V. The Structure of Ultrageodetic Graphs

The goal of this chapter is to provide two different descriptions of ultrageodetic graphs. The first identifies some of their extremal characteristics; one such aspect came to light already in 3.20, which implies that two-connected ultrageodetic graphs are extremal with respect to the inequality $g^* \leq 2 \cdot \text{diam} + 1$ of 3.16.

The first main result, Proposition 5.2, clarifies the extremal position of ultrageodetic blocks among geodetic blocks by providing a sharpened version of 4.18: in a two-connected ultrageodetic graph, each maximal clique is not only the base of a distance-preserving pyramid, but in fact the base of each of a collection of pyramids that covers the vertices of the graph. In the second description, we capitalize on the fact that ultrageodetic graphs share many of the structural features of pyramids. By identifying those features in a series of technical results, we gain intuition about how an ultrageodetic graph "looks", as well as a sufficiently precise description for a characterization of ultrageodetic graphs in Chapter VI.

In preparation for the first main result, here is a technical fact that will be sharpened considerably in 5.3:

5.1 Lemma. Suppose that G is a two-connected graph with no subgraph isomorphic to $K_4 - e$, and that $g^* = 2 \cdot \text{diam} + 1$.

Then every vertex of G lies on a chordless cycle of length g^* .

Proof. If $g^* = 3$, then the assertion is trivial; suppose that $g^* = k > 3$, and let $S \subseteq V$ consist of those vertices that lie on chordless cycles of length k . We wish to show that $S = V$. Suppose, on the contrary, that $V \setminus S$ is non-empty; then there must exist adjacent vertices x_1 in S and y in $V \setminus S$. Consider a chordless cycle $C: x_1 - x_2 - \dots - x_k - x_1$; by 3.15, C preserves distance in G . If y were adjacent to both x_2 and x_k , then y, x_1, x_2 , and x_k would induce a K_4 -e; without loss of generality, suppose that $\{y, x_2\} \notin E$. If y were adjacent to x_i for some i with $2 < i < \text{diam} + 1$, then, for the smallest such i , the locally chordless cycle $y - x_1 - \dots - x_i - y$ would have length $i + 1 \leq \text{diam} + 2 < g^*$ (since $g^* = 2 \cdot \text{diam} + 1 > 3$), in contradiction to the definition of g^* . Thus $\{y, x_i\} \notin E$ for $2 \leq i \leq \text{diam} + 1$. Put $z := x_{\text{diam}+1}$, and let n be the smallest index such that x_n lies on a y, z -geodesic. Since C preserves distance, $d(x_1, z) = \text{diam}$; an application of the triangle inequality shows that

$$\begin{aligned} (1) \quad d(y, z) &\geq d(x_1, z) - d(x_1, y) \\ &= \text{diam} - 1. \end{aligned}$$

Another application of the triangle inequality yields the fact that

$$\begin{aligned} (2) \quad d(y, x_n) &= d(y, z) - d(x_n, z) \\ &= d(y, z) - (\text{diam} + 1 - n) \end{aligned}$$

$$\begin{aligned} &\leq \text{diam} - (\text{diam} + 1 - n) \\ &= n - 1; \end{aligned}$$

similarly,

$$\begin{aligned} (3) \quad d(y, x_n) &= d(y, z) - (\text{diam} + 1 - n) \\ &\geq (\text{diam} - 1) - (\text{diam} + 1 - n) \\ &= n - 2. \end{aligned}$$

Thus $n - 2 \leq d(y, x_n) \leq n - 1$; since $\{y, x_n\} \notin E$, it must be that $n > 2$. Consider a geodesic

$$(4) \quad P: y - v_1 - \dots - v_m - x_n;$$

by (2) and (3),

$$(5) \quad n - 2 \leq m + 1 \leq n - 1.$$

Now the path

$$(6) \quad Q: x_n - x_{n-1} - \dots - x_2 - x_1 - y$$

is chordless, and our choice of x_n ensures that P and Q are internally disjoint; consequently, they determine a DCP, say D , with $\lambda(D) = n + m + 1 \leq 2n - 1$. By 3.13, $\lambda(D) \geq g^\#$, so that

$$(7) \quad 2 \cdot \text{diam} + 1 \leq n + m + 1 \leq 2n - 1.$$

That implies that $n \geq \text{diam} + 1$; hence, $x_n = z$ and $n = \text{diam} + 1$. Then $m = \text{diam} - 1$, and $\lambda(D) = 2 \cdot \text{diam} + 1$. According to 3.13, the canonical cycle PQ is a locally chordless cycle of length $g^\#$; by 3.11, it is chordless. We have a contradiction to the assumption that $y \in V \setminus S$, and we conclude that $S = V$. ■

5.2 Proposition. For a graph G , the following statements

are equivalent:

- i) G is two-connected and ultrageodetic.
- ii) G is two-connected and geodetic, and, for each maximal clique K , and each vertex v in $V(G) \setminus K$, there is a distance-preserving pyramidal subgraph P of G with base K such that $y \in V(P)$ and $\text{diam}(P) = \text{diam}(G)$.
- iii) G has no subgraph isomorphic to $K_4 - e$, and $g^* = 2 \cdot \text{diam} + 1$.

Proof. i) \Rightarrow ii). If G is complete, then there is nothing to prove; suppose that G is not complete. According to i), G is two-connected and ultrageodetic; as we saw in Figure 7, G must also be geodetic. By 4.19, then, G has no extreme points. Let K be a maximal clique, and fix a vertex v in $V(G) \setminus K$. Consider a shortest path

$$(8) \quad x_1 - x_2 - \dots - x_m - v$$

from K to v , and let

$$(9) \quad x_1 - \dots - x_m - v - x_{m+2} - \dots - x_n - y$$

be a geodesic in G that is not T -extendible (it may be that $y = v$). By 3.23, we know that $d(x_1, y) \geq \text{diam}$, so $d(x_1, y) = \text{diam} = e(y)$. Since G is geodetic, we know from 4.3 that the vertex $x_2 = \langle y, x_1, e(y) - 1 \rangle$ is the unique neighbor of x_1 whose distance from y does not exceed $e(y) - 1$; consequently, $d(K, y) = e(y)$, and we can apply 4.20: K is the base of a distance-preserving pyramidal subgraph P of G with apex a such that

$$(10) \quad d(y, K) = d(y, a) + d(a, K).$$

Since P is a subgraph of G , we know that $g^*(P) \geq g^*(G)$; the additional fact that P preserves distance in G implies that $\text{diam}(P) \leq \text{diam}(G)$. As a pyramid, P satisfies the equality $g^*(P) = 2 \cdot \text{diam}(P) + 1$, by 4.16. In summary, we know that

$$(11) \quad g^*(G) \leq g^*(P) = 2 \cdot \text{diam}(P) + 1 \leq 2 \cdot \text{diam}(G) + 1.$$

But the fact that G is ultrageodetic, two-connected, and incomplete implies that $g^*(G) \geq 2 \cdot \text{diam}(G) + 1$, by 3.20. Thus $\text{diam}(P) = \text{diam}(G)$, and it follows that $a = y$; moreover, the vertex v , which lies on the x_1, y -geodesic, must belong to $V(P)$, because P preserves distances in G .

ii) \Rightarrow iii). Since G is two-connected, its augmented girth is finite. The fact that G is geodetic implies that g^* is odd, by 4.2, and that G has no subgraph isomorphic to $K_4 - e$, by 4.1. Suppose that $g^* = 2k + 1$. If $k = 1$, then G is a block graph, by 3.14, and thus complete, by 1.4; it follows that $\text{diam}(G) = 1$. Suppose, then, that $k > 1$, and let C be a chordless cycle of length g^* . Fix a vertex $x \in V(C)$, and let y and z be vertices opposite x on C . Put $K := M(y, z)$, and let P be a distance-preserving pyramidal subgraph of G with base K such that $x \in V(P)$ and $\text{diam}(P) = \text{diam}(G)$. By 3.15, C is a distance-preserving subgraph, and thus consists of the geodesics joining x , y , and z ; consequently, it must be a subgraph of the distance-preserving subgraph P . It follows from 4.16 that

$$\begin{aligned}
 (12) \quad g^*(G) &= 2k + 1 \\
 &= \lambda(C) \\
 &\geq g^*(P) \\
 &= 2 \cdot \text{diam}(P) + 1 \\
 &= 2 \cdot \text{diam}(G) + 1.
 \end{aligned}$$

But $g^*(G) \leq g^*(P)$, since P is a subgraph of G , and we conclude that $g^*(G) = 2 \cdot \text{diam}(G) + 1$.

iii) \Rightarrow i). Suppose that G has no subgraph isomorphic to $K_4 - e$, and that $g^* = 2 \cdot \text{diam} + 1$. By 3.20, G is ultrageodetic, so we need only show that G is two-connected. To that end, let B be a block of G such that $g^*(B) = g^*(G)$. If $V(B) \neq V(G)$, then there must exist adjacent vertices x in $V(B)$ and y in $V(G) \setminus V(B)$. By Lemma 5.1, since B is two-connected, x lies on a chordless cycle C of length $g^*(G)$; Corollary 3.15 states that C preserves distance in G . Choose a vertex $z \in V(C)$ such that $d(x, z) = \text{diam}$. Since $y \in V(B)$, it must be that

$$(13) \quad d(y, z) = d(y, x) + d(x, z) = 1 + \text{diam};$$

but that is nonsense, and we conclude that $V(B) = V(G)$, i. e., that G is two-connected. ■

5.3 Corollary. The following conditions are necessary and sufficient for a two-connected graph G to be ultrageodetic:

- i) G has no subgraph isomorphic to $K_4 - e$.
- ii) g^* is odd.
- iii) Each pair of vertices of G lies on a chordless

cycle of length g^* .

Proof. The necessity of the three conditions follows immediately from 5.2, so we need only demonstrate their sufficiency. Suppose that G satisfies all three conditions. If $g^* = 3$, then i) and iii) imply that G is a block graph, by 3.14, and therefore ultrageodetic, by 3.20. Suppose that $g^* > 3$; by ii), then, $g^* > 4$, so 3.15 and i) imply that each chordless cycle of length g^* preserves distance in G . Thus, for vertices x and y with $d(x, y) = \text{diam}$, the chordless cycle whose existence is ensured by iii) must have length at least $2 \cdot \text{diam}$; ii) and 3.16 imply that its length is odd and no more than $2 \cdot \text{diam} + 1$, respectively. It follows that $g^* = 2 \cdot \text{diam} + 1$, and we conclude that G is ultrageodetic, by 5.2. ■

According to 1.6, a graph in which every block is geodetic must itself be geodetic. It is not so easy to build new ultrageodetic graphs from old ones:

5.4 Proposition. If an ultrageodetic graph has a cut-vertex, then it is a block graph.

Proof. Suppose that G is an ultrageodetic graph with a cut-vertex. If G is a tree, then it is a block graph, by 1.4; suppose, then, that G has a cycle, and let B be a block of G with $g^*(B) = g^*(G)$. On the one hand, since G is not two-connected, $\text{diam}(B) < \text{diam}(G)$; on the other hand, since B is two-connected, 3.3 implies that $g^*(B) = 2 \cdot \text{diam}(B) + 1$. It follows that

$$(14) \quad g^*(G) = g^*(B) = 2 \cdot \text{diam}(B) + 1 < 2 \cdot \text{diam}(G) + 1,$$

so we deduce from 3.20 that G is a block graph. ■

Since can identify both ultrageodetic graphs having cut-vertices as block graphs, and two-connected graphs having extreme points as complete graphs, by 4.19, we shall no longer consider either possibility:

5.5 Standing hypothesis. Throughout the remainder of this investigation, we assume that G is an incomplete ultrageodetic block; that is, G is two-connected and ultrageodetic, and it has no extreme points.

In the succeeding lemmata, we see that pyramids model ultrageodetic graphs in general quite nicely, in this sense: for each maximal clique K , we may picture the graph as a "pyramid" (see Figure 13) that is "suspended" (see the discussion preceding 4.20) from its base K . In place of the vertices on the suspended path from a vertex $x \in K$ to the apex, we see the sets $R(K, x; k)$; the "apex", then, is the sphere $S(K; \text{diam})$. Alternatively, we may view the graph as a "pyramid" that is "suspended" from a single vertex x (see Figure 14). Now x is the "apex", and the "base" is $S(x; \text{diam})$, which need not be a clique; as before, the other "vertices" are k -reaches -- in this case, the sets $R(x, y; k)$ for neighbors y of x . Each of the next five results can be viewed as the verification that some aspect of Figure 13 or Figure 14 is valid; however, we make no use of the figures in any of the proofs. The first result in this vein indicates that a geodesic joining a vertex in the "base" of Figure 13 to a vertex outside the "base" lies as

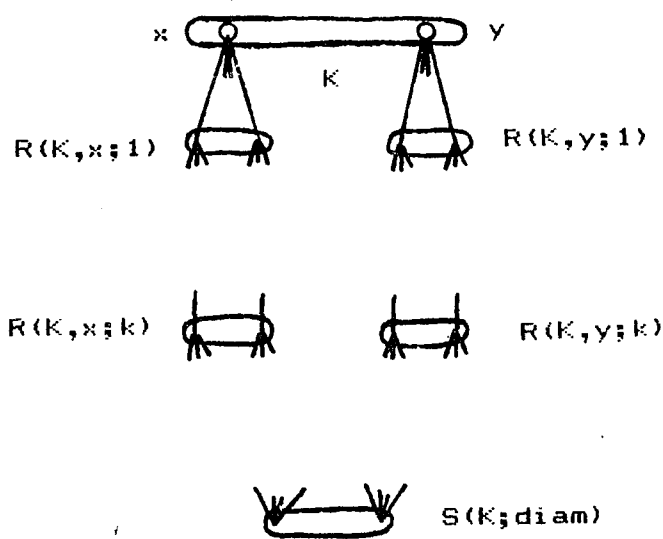


Figure 13.

Suspension of an ultrageodetic graph from a maximal clique.

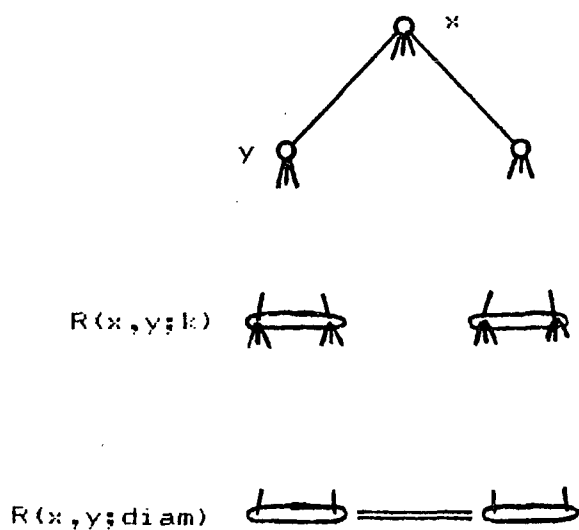


Figure 14.

Suspension of an ultrageodetic graph from a vertex.

it would in a pyramid.

5.6 Lemma. For adjacent vertices x and y , and the clique $K := M(x, y)$, if w lies in $R(K, y; k)$, then $k < \text{diam}$ iff $y = \langle x, w; 1 \rangle$.

Proof. On the one hand, if $y = \langle x, w; 1 \rangle$, then

$$(15) \quad k = d(y, w) = d(x, w) - 1 \leq \text{diam} - 1.$$

On the other hand, if $k < \text{diam}$, then the chordless path

$$(16) \quad x - y - \langle y, w; 1 \rangle - \dots - \langle y, w; k-1 \rangle - w$$

belongs to \mathcal{T} . As G is ultrageodetic and $\Gamma \in \mathcal{T}$, that path is the x, w -geodesic; consequently, $y = \langle x, w; 1 \rangle$. ■

For $k < \text{diam}$, the sets $R(K, x; k)$ and $R(K, y; k)$ of Figure 13 appear to be disjoint, and there is no indication of edges between them. We now show that aspect of the illustration to be correct.

5.7 Lemma. For adjacent vertices x and y , and the maximal clique $K := M(x, y)$, $R(K, x) \cap R(K, y) = S(K; \text{diam})$. Moreover, if $v \in R(K, x)$ is adjacent to $w \in R(K, y)$, then either $v = x$ and $w = y$, or at least one of v and w lies in $S(K; \text{diam})$.

Proof. We begin with the inclusion " \supseteq ": suppose that z lies in $S(K; \text{diam})$. For every $w \in K$, then, we see that

$$(17) \quad \text{diam} \geq d(w, z) \geq d(K, z) = \text{diam};$$

consequently, $z \in R(K, w; \text{diam}) \subseteq R(K, w)$, and it follows that

$$(18) \quad z \in \bigcap \{R(K, w); w \in K\}.$$

For the inclusion " \subseteq ", fix $z \in R(K, x) \cap R(K, y)$, and

consider a distance-preserving pyramidal subgraph P with base K such that $z \in V(P)$ and $\text{diam}(P) = \text{diam}(G)$; by 5.2, such a pyramid exists. Since K , x , y , and z all lie in P , the fact that P preserves distance implies that

$$(19) \quad z \in R_P(K, x) \cap R_P(K, y);$$

by 4.16, then, z is the apex of P . That is,

$$(20) \quad d(z, K) = \text{diam}(P) = \text{diam}(G),$$

and we conclude that $z \in S(K; \text{diam})$.

Now suppose that $v \in R(K, x; m)$ and $w \in R(K, y; n)$ are adjacent for some $m, n \geq 0$. From the preceding argument, we know that $R(K, x; j) \cap R(K, y; j) = \emptyset$ for each $j < \text{diam}$; hence, for each $j < \text{diam}$, $\cap \{r(K, w; j); w \in K\} = \emptyset$, and it follows from 4.11 that no vertex in $R(K, x; j)$ has a neighbor in $R(K, y; j)$. In particular, if $m = n$, then $m = \text{diam}$ or $m = 0$; in the former case, v and w lie in $S(K; \text{diam})$, and in the latter, $v = x$ and $w = y$. Thus, without loss of generality, we may assume that $m < n$. By the definitions and the triangle inequality, it follows that

$$\begin{aligned} (21) \quad n &= d(K, w) \\ &\leq d(x, w) \\ &\leq d(x, v) + d(v, w) \\ &= m + 1 \\ &\leq n; \end{aligned}$$

then equality holds throughout, and $d(x, w) = n$. Therefore, w belongs to $R(K, x; n) \cap R(K, y; n)$, and we conclude from the first part of the Lemma that $w \in S(K; \text{diam})$. ■

For vertices x and y in a geodetic graph, we found in 4.4 that there is a unique maximal clique in the neighborhood of y that is closer to x than y is, namely, $M(y,x)$; for ultrageodetic graphs, the remaining vertices in the neighborhood have a particularly nice form:

5.8 Lemma. Let x and y be distinct vertices such that $d(x,y) = k < \text{diam}$. Then $B(y;1) = R(x,y;k+1) \cup M(y,x)$.

Proof. We establish first that the union is in fact disjoint. By 4.6, the vertex $w := \langle y,x;1 \rangle = \langle x,y;k-1 \rangle$ lies in $M(y,x)$. Thus, on the one hand, if $z \in M(y,x)$, then

$$(22) \quad d(z,x) \leq d(z,w) + d(w,x) = k;$$

on the other hand, if $z \in R(x,y;k+1)$, then

$$(23) \quad d(z,x) = k + 1.$$

The inclusion " \supseteq " follows immediately from the definitions. For the inclusion " \subseteq ", fix $z \in B(y;1) \setminus M(y,x)$; then

$$(24) \quad d(z,x) \leq d(z,y) + d(y,x) = k + 1.$$

Consider the maximal clique $K = M(z,y)$; for each $v \in K$, since $\langle y,x;1 \rangle \in M(y,x)$ is the unique neighbor of y whose distance from x does not exceed $k - 1$ (by 4.3), it must be that $d(v,x) \geq k$. Consequently, $d(K,x) = k$. But $k < \text{diam}$, so $\bigcap \{R(K,v;k); v \in K\} = \emptyset$, by 4.11; in particular, according to 4.8,

$$(25) \quad R(K,y;k) \cap R(K,z;k) \subseteq R(K,y) \cap R(K,z) \\ = S(K;\text{diam}),$$

by 5.7. But $k < \text{diam}$, so $R(K,y;k) \cap R(K,z;k) = \emptyset$. Consequently, $d(z,x) > k$; that is,

$$(26) \quad d(z,x) = k + 1 = d(z,y) + d(y,x),$$

and $z \in R(x,y;k+1)$. ■

5.9 Lemma. For each vertex x in a maximal clique K ,
 $S(K;\text{diam}) = R(K,x;\text{diam})$.

Proof. The sphere $S(K;\text{diam})$ consists of all the vertices v whose distance from K is diam ; by definition, the distance from such a vertex v to an element x of K must be at least diam , but of course it cannot exceed diam . It follows that the sphere is included in the reach; the reverse inclusion is an immediate consequence of the definition of the reach. ■

Now we turn our attention to various facets of Figure 14. The existence of nonadjacent neighbors y and z of a vertex x is ensured by our assumption that x is not an extreme point (5.5 and 2.11).

5.10 Lemma. Suppose that y and z are nonadjacent neighbors of a vertex x . Then, for each $v \in R(x,y;\text{diam})$,
 $\langle v,z;1 \rangle \in R(x,z;\text{diam})$.

Proof. Fix $v \in R(x,y;\text{diam})$; then the vertex

$$(27) \quad y = \langle x,v;1 \rangle = \langle v,x;\text{diam}-1 \rangle$$

is the unique neighbor of x whose distance from v does not exceed $\text{diam} - 1$, by 4.3. Thus, for each neighbor $t \neq y$ of x , $d(t,v) = \text{diam}$. For the maximal clique $K := M(x,z)$, the preceding argument and the fact that $d(x,v) = \text{diam}$ imply together that $d(K,v) = \text{diam}$. Consider

now the vertices

$$(28) \quad u := \langle v, x; 1 \rangle = \langle x, v; \text{diam}-1 \rangle$$

and

$$(29) \quad w := \langle v, z; 1 \rangle = \langle z, v; \text{diam}-1 \rangle.$$

By the triangle inequality and the definitions, we see that

$$\begin{aligned} (30) \quad \text{diam} &= d(x, v) \\ &= d(x, u) + d(u, v) \\ &\geq d(K, u) + d(u, v) \\ &\geq d(K, v) \\ &= \text{diam}; \end{aligned}$$

similarly,

$$\begin{aligned} (31) \quad \text{diam} &= d(z, v) \\ &= d(z, w) + d(w, v) \\ &\geq d(K, w) + d(w, v) \\ &\geq d(K, v) \\ &= \text{diam}. \end{aligned}$$

Then equality holds in both (30) and (31), and we see that

$$(32) \quad d(K, w) = d(K, u) = \text{diam} - 1;$$

furthermore,

$$(33) \quad d(K, u) = d(x, u) = d(K, x) + d(x, u)$$

and

$$(34) \quad d(K, w) = d(z, w) = d(K, z) + d(z, w).$$

By 1.2, those equalities imply that $u \in R(K, x)$ and that $w \in R(K, z)$, respectively. According to 4.9 and 5.7,

$$(35) \quad R(K, x) \cap R(K, z) = \cap \{R(K, t); t \in K\} = S(K, \text{diam}).$$

Thus, since u and w do not belong to $S(K; \text{diam})$, it must be that $u \neq w$. Now $u = \langle x, v; \text{diam}-1 \rangle$ is the unique neighbor of v whose distance from x is less than diam ; since w is also a neighbor of v , then, $d(w, x) = \text{diam}$. Because $d(w, z) = \text{diam} - 1$, we conclude that w lies in $R(x, z; \text{diam})$, as asserted. ■

As we shall see now, the "base" of Figure 14 is not a clique unless $r(x, v; 1) = 1$ for each neighbor v of x .

5.11 Lemma. Suppose that y and z are nonadjacent neighbors of a vertex x . Then the edges between $R(x, y; \text{diam})$ and $R(x, z; \text{diam})$ form a perfect matching.

Proof. Fix a vertex $v \in R(x, y; \text{diam})$; according to 4.3, $y = \langle v, x; \text{diam}-1 \rangle$ is the unique neighbor of x whose distance from v does not exceed $\text{diam} - 1$, so it must be that $d(v, z) = \text{diam}$. Due to the fact that every member of $R(x, z; \text{diam})$ is at distance $\text{diam} - 1$ from z , Lemma 4.3 implies that v can be adjacent to at most one of them. Lemma 5.10 states that it is adjacent to one, specifically, $\langle v, z; 1 \rangle$. Repeating that argument with the roles of y and z interchanged, we see that each vertex in one of $R(x, y; \text{diam})$ and $R(x, z; \text{diam})$ has exactly one neighbor in the other. ■

In the last of our technical lemmata, we examine the relationship between the reaches of a clique and the reaches of its vertices:

5.12 Lemma. Let x and y be distinct vertices in a maxi-

mal clique K , and fix nonnegative integers j and k such that $j < \text{diam}$ and $j \leq k \leq \text{diam}$. Then, for each vertex v in $R(K, y; j)$, $R(x, v; k) = R(K, v; k-1)$. In particular, $R(x, y; k) = R(K, y; k-1)$.

Proof. Fix $v \in R(K, y; j)$. Since

$$(36) \quad R(K, y; j) \cap R(K, x; j) \subseteq S(k; j) \cap R(k, y) \cap R(K, x),$$

the fact that $j < \text{diam}$ implies that $R(K, y; j) \cap R(K, x; j)$ is empty, by 5.7. Specifically, $v \notin R(K, x; j)$. But $d(v, K) = j$, so it must be that $d(x, v) = j + 1$; consequently, $y = \langle x, v; 1 \rangle$. Suppose that z lies in $R(x, v; k)$; then $d(z, v) = k - j - 1$. Furthermore, by the triangle equalities of 1.1 and 4.6, together with two applications of the triangle inequality, we see that

$$\begin{aligned} (37) \quad d(x, z) &= d(x, v) + d(v, z) \\ &= d(x, y) + d(y, v) + d(v, z) \\ &\geq d(x, y) + d(y, z) \\ &\geq d(x, z). \end{aligned}$$

The equality of the first and last terms of (37) implies that equality holds throughout, so that

$$(38) \quad d(y, z) = d(x, z) - d(x, y) = k - 1;$$

moreover, $y = \langle x, z; 1 \rangle$ is the unique neighbor of x whose distance from z does not exceed $k - 1$, by 4.3. It follows that $d(K, z) = k - 1$; since we already know that $d(z, v) = k - j - 1$ and $d(K, v) = j$, we see that z belongs to $R(K, v; k-1)$. Now suppose that z lies in $R(K, v; k-1)$. Then $d(v, z) = k - 1 - j$; that equality and the fact that

$d(y,v) = j$ imply that $d(y,v) + d(v,z) = d(K,z)$. Applying the triangle inequality and the fact that $d(y,z) \geq d(K,z)$, we see that

$$\begin{aligned} (39) \quad d(y,z) &\leq d(y,v) + d(v,z) \\ &= d(K,z) \\ &\leq d(y,z). \end{aligned}$$

The equality of the first and last terms implies that equality holds throughout (39); specifically, $d(y,z) = d(K,z)$ and $d(y,z) = d(y,v) + d(v,z)$. Therefore, $z \in R(K,y;k-1)$; since $k-1 < \text{diam}$, it follows as before from 5.7 that $R(K,y;k-1) \cap R(K,x;k-1) = \emptyset$. In particular, z does not belong to $R(K,x;k-1)$, and we deduce that $d(x,z) \geq k$. Now we compute that

$$\begin{aligned} (40) \quad k &\leq d(x,z) \\ &\leq d(x,y) + d(y,z) \\ &= 1 + (k-1). \end{aligned}$$

The equality of the first and last terms implies that $d(x,z) = k$; but $d(x,v) = j+1$ and $d(v,z) = k-j-1$, so z lies in $R(x,v;k)$. We have seen that $z \in R(x,v;k)$ iff $z \in R(K,v;k-1)$. The special case now follows if we set $j := 0$. ■

Having taken care of the technical preliminaries, we address the following question: what restrictions are imposed on the degrees of vertices by the fact that the graph is ultrageodetic? Our first answer to that question, in Proposition 5.16, is based on the fact that spheres of radius diam are "balanced":

5.13 Proposition. For adjacent vertices x and y ,

$$r(x,y;\text{diam}) = s(x;\text{diam})/\delta(x).$$

Proof. In light of the fact that

$$(41) \quad S(x;\text{diam}) = \bigcup \{R(x,w;\text{diam}); w \in S(x;1)\}$$

(the union is disjoint by 4.8), we just need to establish that $r(x,y;\text{diam}) = r(x,z;\text{diam})$ for each pair of neighbors y,z of a vertex x . For nonadjacent y and z , that is an immediate consequence of 5.11. Suppose that y and z are adjacent, and set $K := M(y,z)$; observe that $x \in K$. Since x is not an extreme point, it must have a neighbor $w \notin K$, by 2.11; since there is no subgraph isomorphic to $K_4 - e$, it follows from 1.5 that x is the only neighbor of w in K . Now two applications of 5.11 yield the equality

$$(42) \quad r(x,y;\text{diam}) = r(x,w;\text{diam}) = r(x,z;\text{diam}). \quad \blacksquare$$

5.14 Definition. For each vertex x of an incomplete ultrageodetic graph, $r(x) := s(x;\text{diam})/\delta(x)$.

5.15 Lemma. For adjacent vertices x and y ,

$$(43) \quad \begin{aligned} S(x;\text{diam}) \setminus R(x,y;\text{diam}) &= S(x;\text{diam}) \cap S(y;\text{diam}) \\ &= S(y;\text{diam}) \setminus R(y,x;\text{diam}). \end{aligned}$$

Proof. Fix $w \in S(x;\text{diam})$; by the triangle inequality,

$$(44) \quad d(y,w) \geq d(x,w) - d(x,y) = \text{diam} - 1,$$

so w lies in $S(y;\text{diam})$ iff it does not lie in $R(x,y;\text{diam})$. Since the roles of x and y may be interchanged, both equalities follow. \blacksquare

The partitioning of spheres of radius diam into classes of equal size, as described in 5.12, yields a near-

regularity for degrees:

5.16 Proposition. For each pair of vertices x and y ,
 $r(x) \cdot (\delta(x) - 1) = r(y) \cdot (\delta(y) - 1)$.

Proof. Since the graph is connected, it suffices to prove the assertion for adjacent vertices x and y . In that case, by 5.15,

$$S(x; \text{diam}) \setminus R(x, y; \text{diam}) = S(y; \text{diam}) \setminus R(y, x; \text{diam});$$

counting elements on each side with the aid of 5.13, we find that

$$(45) \quad r(x) \cdot d(x) - r(x) = r(y) \cdot d(y) - r(y),$$

as desired. ■

Although ultrageodetic graphs need not be regular (e.g., pyramids are not regular), 5.16 can be sharpened for certain vertices:

5.17 Proposition. Suppose that the vertices x , y , and z all lie in a maximal clique K . Then $r(x) = r(y) = r(z)$ and $\delta(x) = \delta(y) = \delta(z)$.

Proof. On the one hand, by 5.13 and the definition of $r(\cdot)$, we see that

$$(46) \quad r(x) = r(x, z; \text{diam}) \quad \text{and} \quad r(y) = r(y, z; \text{diam});$$

on the other hand, we know that

$$(47) \quad r(x, y; \text{diam}) = r(K, y; \text{diam}) = r(z, y; \text{diam}),$$

by 5.12. It follows that $r(x) = r(y)$, so that 5.16 yields the equality

$$(48) \quad r(x) \cdot (\delta(x) - 1) = r(z) \cdot (\delta(z) - 1) = r(x) \cdot (\delta(z) - 1);$$

consequently, $\delta(x) = \delta(z)$. By interchanging the roles of y and z , we deduce that $\delta(x) = \delta(y)$ and $r(x) = r(y)$. ■

In order to simplify the counting arguments in the remaining results, we introduce the following terminology:

5.18 Definition. For distinct vertices x and y of an incomplete ultrageodetic block, we define the relative degree of x with respect to y to be

$$(49) \quad \delta(x, y) := \delta(x) - m(x, y) + 2.$$

(Recall that $m(x, y) = |M(x, y)|$; see 4.5.)

5.19 Lemma. If $v = \langle x, y; k \rangle$ for some k , then

$$\delta(x, v) = \delta(x, y).$$

Proof. Suppose that $v = \langle x, y; k \rangle$ for some k , and consider the maximal clique $K := M(x, y)$; then $d(K, y) = d(x, y) - 1$, and $w := \langle x, y; 1 \rangle \in K$, by 4.6. Moreover, the definition of v and the triangle inequality yield that

$$\begin{aligned} (50) \quad d(x, y) &= d(x, v) + d(v, y) \\ &= d(x, \langle x, v; 1 \rangle) + d(\langle x, v; 1 \rangle, v) + d(v, y) \\ &\geq d(x, \langle x, v; 1 \rangle) + d(\langle x, v; 1 \rangle, y) \\ &\geq d(x, y). \end{aligned}$$

Since the first and last terms are identical, equality must hold throughout (50); the resulting triangle equality

$$(51) \quad d(x, y) = d(x, \langle x, v; 1 \rangle) + d(\langle x, v; 1 \rangle, y)$$

implies by 4.6 that $\langle x, v; 1 \rangle = \langle x, y; 1 \rangle = w$. That is, w is the unique neighbor of x whose distance from v is less than $d(x, v)$; thus $M(x, v) = K = M(x, y)$, and the assertion follows from the definition of $\delta(-, -)$. ■

5.20 Proposition. Let x and y be distinct vertices such that $d(x,y) = k < \text{diam}$. Then $r(x,y;k+1) = \delta(y,x) - 1$.

Proof. According to 5.8,

$$(52) \quad B(y;1) = R(x,y;k+1) \cup M(y,x).$$

By counting elements on each side, we see that

$$(53) \quad \delta(y) + 1 = r(x,y;k+1) + m(y,x),$$

and the assertion follows immediately. ■

5.21 Proposition. Let x and y be vertices such that $d(x,y) = \text{diam}$. Then $\delta(x,y) = \delta(y,x)$.

Proof. We shall prove that $\delta(x,y) \geq \delta(y,x)$; since the roles of x and y may be interchanged, the assertion follows. Consider the maximal clique $K := M(x,y)$ and the vertex $z := (x,y;1)$; by 4.6, $z \in K$. Our choice of K and z ensure that

$$(54) \quad d(K,y) = d(x,y) - 1 = d(z,y);$$

hence, according to 4.22,

$$(55) \quad R(K,y;\text{diam}) = R(z,y;d(z,y) + 1).$$

By counting elements on each side of (55), and applying 5.20 and 5.19, we see that

$$\begin{aligned} (56) \quad r(K,y;\text{diam}) &= r(z,y;d(z,y)+1) \\ &= \delta(y,z) - 1 \\ &= \delta(y,x) - 1. \end{aligned}$$

On the other hand, since $x \in K$, each neighbor of x lies either in K or in $R(K,x;1)$; that is,

$$(57) \quad K \cup R(K,x;1) = B(x;1).$$

Using the fact that $K = M(x, y)$, we count elements on each side of (57), obtaining that

$$(58) \quad m(x, y) + r(K, x; 1) = \delta(x) + 1,$$

and thus also that

$$(59) \quad r(K, x; 1) = \delta(x, y) - 1.$$

Combining (58) and (59), we see that it will suffice to show that

$$(60) \quad r(K, y; \text{diam}) \leq r(K, x; 1).$$

To that end, we define the function

$$(61) \quad f: R(K, y; \text{diam}) \rightarrow R(K, x; 1)$$

by means of the assignment

$$(62) \quad w \mapsto \langle x, w; 1 \rangle;$$

to verify that $\langle x, w; 1 \rangle$ belongs to $R(K, x; 1)$, we observe that

$$(63) \quad R(K, y; \text{diam}) = S(K; \text{diam}) = R(K, x; \text{diam}),$$

by 5.9. Since $w \in R(K, y; \text{diam})$, it also lies in $R(K, x; \text{diam})$. According to 4.21, then, $d(K, \langle x, w; 1 \rangle) = 1$, and it follows that $\langle x, w; 1 \rangle \in R(K, x; 1)$. To conclude that $r(K, y; \text{diam}) \leq r(K, x; 1)$, we need only know that f is injective; suppose that it is not, and fix distinct vertices $u, w \in R(K, y; \text{diam})$ such that

$$(64) \quad f(u) = f(w) =: v.$$

Then u and w are both neighbors of y at distance $\text{diam} - 1$ from v , so it follows from 4.3 that $d(y, v) < \text{diam}$; but z is the unique neighbor of x whose

distance from y is less than diam , and we reach a contradiction. Therefore, f is injective, as desired. ■

Our first application of 5.21 is to demonstrate the "inverted symmetry" of the reaches of a maximal clique:

5.22 Proposition. Let x and y be adjacent vertices, and put $K := M(x, y)$; for vertices v in $R(x, y; k)$ and w in $R(y, x; \text{diam} + 1 - k)$, with $k \geq 1$,

$$r(K, v; k) = r(K, w; \text{diam} + 1 - k).$$

Proof. Fix $v \in R(x, y; k)$ and $w \in R(y, x; \text{diam} + 1 - k)$, and let P and Q be the v, y -geodesic and the x, w -geodesic, respectively. Since

$$(65) \quad V(P) \subseteq R(K, y) \quad \text{and} \quad V(Q) \subseteq R(K, x),$$

it follows from 5.7 that $(Px)Q$ is a path. By 4.11, $(Px)Q$ is chordless; its length is $\lambda(P) + \lambda(Q) + 1 = \text{diam}$, so it is the unique v, w -path in T , and therefore a geodesic, since $\Gamma \subseteq T$. Consequently, $d(v, w) = \text{diam}$. Furthermore, as x and y both lie on the v, w -geodesic $(Px)Q$, we see that $d(v, w) = d(v, y) + d(y, w)$ and $d(w, v) = d(w, x) + d(x, v)$; that is, $y = \langle v, w; k-1 \rangle$ and $x = \langle w, v; \text{diam}-k \rangle$. According to 5.19, then, $\delta(v, y) = \delta(v, w)$ and $\delta(w, x) = \delta(w, v)$; but $d(v, w) = \text{diam}$, so 5.21 implies that $\delta(v, w) = \delta(w, v)$, and it follows that

$$(66) \quad \delta(v, y) = \delta(v, w) = \delta(w, v) = \delta(w, x).$$

By 5.12, $R(K, y; k-1) = R(x, y; k)$; since v lies in the former set, 4.22 implies that $R(K, v; k) = R(y, v; k)$; similarly, $R(K, w; \text{diam} + 1 - k) = R(x, w; \text{diam} - k + 1)$. Therefore, by 5.20,

$$(67) \quad r(K, v; k) = r(y, v; k) = \delta(v, y) - 1, \text{ and}$$

$$(68) \quad r(K, w; \text{diam}+1-k) = r(x, w; \text{diam}+1-k) = \delta(w, x) - 1.$$

Combining (67) and (68) with the aid of (66), we see that

$$(69) \quad r(K, v; k) = r(K, w; \text{diam}-k+1). \quad \blacksquare$$

The symmetry described in 5.22 is rather awkward, and it would be pleasant if the reaches of a maximal clique K through two vertices in the same sphere about K were equipotent. Although that is not the case in general, it is true in two special situations; in each case, the proof is a simple application of 5.22:

5.23 Corollary. Let x be a vertex lying in a maximal clique K , and suppose that the vertices u and v both lie in $R(K, x; k-1)$ for some k . Then $r(K, u; k) = r(K, v; k)$.

Proof. Fix a vertex $y \neq x$ in K ; by 5.12,

$$(70) \quad R(K, y; \text{diam}-k) = R(x, y; \text{diam}+1-k) =: A.$$

Since u and v lie in $R(y, x; k)$, two applications of 5.22 yield the following equality for each vertex $w \in A$:

$$(71) \quad r(K, u; k) = r(K, w; \text{diam}+1-k) = r(K, v; k). \quad \blacksquare$$

5.24 Corollary. For a nontrivial maximal clique K , and vertices v and w such that $d(v, K) = d(w, K) = k - 1$, $r(K, v; k) = r(K, w; k)$.

Proof. Since $R(K, x; k) = \emptyset$ for $k > \text{diam}$, suppose that $k \leq \text{diam}$. For some $x, y \in K$ (maybe $x = y$), v lies in $R(K, x; k-1)$ and w lies in $R(K, y; k-1)$. Since K is nontrivial, there exists a $z \in K \setminus \{x, y\}$. Applying 5.12 twice, we see that

$$(72) \quad R(x, z; \text{diam}+1-k) = R(K, z; \text{diam}-k) = R(y, z; \text{diam}+1-k);$$

consequently, for each u lying in

$R(K, z; \text{diam} - k)$, 5.22 implies that

$$(73) \quad \begin{aligned} r(K, v; k) &= r(K, u; \text{diam}+1-k) \\ &= r(K, w; k). \quad \blacksquare \end{aligned}$$

As we shall see now, the preceding result provides a means for counting the vertices in arbitrary reaches $R(K, v; k)$ of a nontrivial maximal clique K , whether or not v lies in $S(K; k-1)$:

5.25 Definition. For a nontrivial maximal clique K , and each positive integer $k \leq \text{diam}$, set $r(K; k-1, k) := r(K, v; k)$ for some $v \in S(K; k-1)$; by 5.24, the particular choice of v has no effect on that value. For integers i, j such that $0 \leq i < j \leq \text{diam}$, we define $r(K; i, j)$ as follows:

$$(74) \quad r(K; i, j) := \prod_{k=i+1}^j r(K; k-1, k).$$

5.26 Proposition. Let K be a nontrivial maximal clique. For integers i and j such that $0 \leq i < j \leq \text{diam}$, if v lies in $S(K; i)$, then $r(K, v; j) = r(K; i, j)$.

Proof. Fix an integer i such that $0 \leq i < \text{diam}$ and a vertex $v \in S(K; i)$; we shall prove by induction that the assertion holds for every j such that $i + 1 \leq j \leq \text{diam}$. For $j = i + 1$, the assertion follows immediately from the definition: $r(K, v; i+1) = r(K; i, i+1)$. Suppose that the statement holds for some j such that $i < j < \text{diam}$; then

$r(K, v; j) = r(K; i, j)$. According to 4.3, each vertex x in $R(K, v; j+1)$ has a unique neighbor in $R(K, v; j)$, namely, $\langle x, v; 1 \rangle$. That is, each vertex $x \in R(K, v; j+1)$ lies in $R(K, w; j+1)$ for exactly one $w \in R(K, v; j)$, and we see that

$$(75) \quad R(K, v; j+1) = \bigcup \{R(K, w; j+1); w \in R(K, v; j)\}.$$

By 5.25, $r(K, w; j+1) = r(K; j, j+1)$ for each vertex w in $R(K, v; j)$; thus, as we already know that $r(K, v; j) = r(K; i, j)$, we obtain the desired equality:

$$(76) \quad r(K, v; j+1) = r(K; i, j) \cdot r(K; j, j+1) = r(K; i, j+1). \quad \blacksquare$$

Chapter VI. A Classification of Ultrageodetic Blocks

We continue here the description of ultrageodetic blocks begun in Chapter 5, albeit with a somewhat different strategy. Primarily, we have presented up to this point structural properties common to all ultrageodetic blocks, irrespective of the diameter or the existence of nontrivial maximal cliques (the exceptions were 5.17 and 5.24 - 5.26). Now we take into account both the diameter and the number of trivial and nontrivial maximal cliques containing certain vertices. Our goal is to extend the Stemple-Kantor classification 1.7 to ultrageodetic graphs of diameter greater than two. Proposition 1.7 was stated for geodetic graphs of diameter two; however, since each chordless path of length one or two is a geodesic (in every graph), a graph with diameter two is ultrageodetic if and only if it is geodetic. For diameters greater than two, the fundamental fact is 6.6: a vertex lying in both a trivial maximal clique and a nontrivial maximal clique lies in exactly one of each. That fact indicates that ultrageodetic blocks with diameter greater than two differ structurally from those of diameter two (see Figure 15). However, it facilitates the proof in 6.13 that ultrageodetic blocks of large diameter share the properties set forth in 1.7 for those with diameter two. Finally, we prove in 6.21 that there are at most three kinds of ultrageodetic graphs of diameter greater than two: pyra-

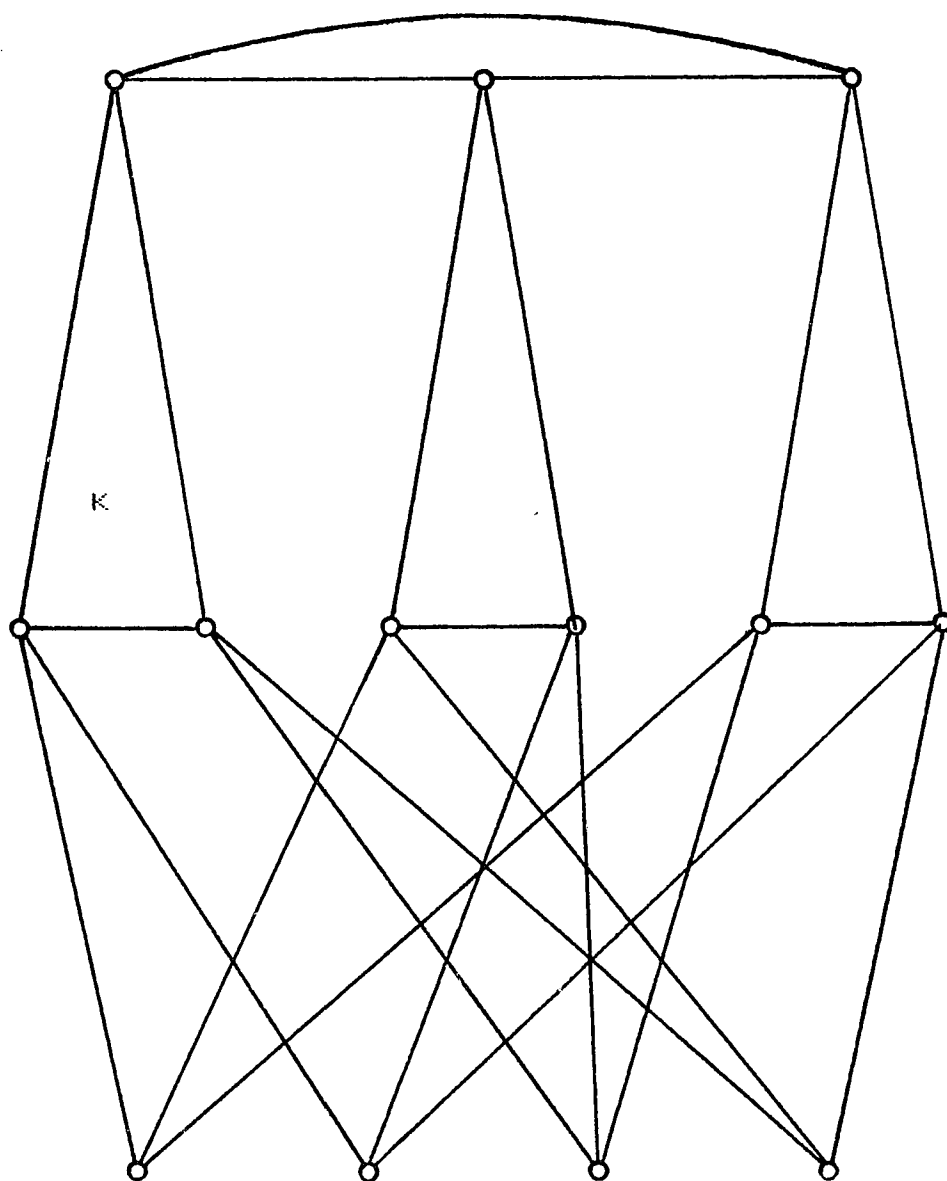


Figure 15.

A geodetic graph of diameter two with vertices lying in a nontrivial maximal clique K and two trivial maximal cliques.

mids, Moore geometries, and certain graphs associated with finite projective planes, for which we give an explicit construction. In view of Proposition 1.12, it follows that every ultrageodetic graph is a pyramid or one of the graphs described in 6.21. Throughout the chapter,

Standing Hypothesis 5.5 remains in effect!

We begin by identifying which ultrageodetic graphs are Moore graphs or thick Moore geometries.

6.1 Proposition. The following four conditions are equivalent:

- i) Every maximal clique is trivial.
- ii) $g = g^*$.
- iii) G is a Moore graph.
- iv) G is strongly geodetic.

Moreover, if any one of i) – iv) holds, then G is regular.

Proof. Since triangles are nontrivial maximal cliques, the equivalence of i) and ii) follows immediately from the definitions of g and g^* . The equivalence of ii) and iii) follows from 1.10 and 5.2: $g^* = 2 \cdot \text{diam} + 1$.

The equivalence of iii) and iv) is Proposition 1.11.

(Alternatively, the equivalence of ii) and iv) follows from 5.2 and 3.6.)

Suppose now that any one of i) – iv) holds; then all of them hold, by the preceding argument. To show that G is regular, it suffices to prove that the degrees of adjacent vertices are equal. Let x and y be adjacent vertices, and consider the apex a of a distance-preserving pyramid

with base $K := M(x, y)$ and diameter equal to that of G (by 5.2, such a pyramid exists). According to 5.21, the fact that $d(x, a) = \text{diam} = d(y, a)$ implies that

$$(1) \quad \delta(x, a) = \delta(a, x) \quad \text{and} \quad \delta(y, a) = \delta(a, y).$$

For each pair of distinct vertices z, w , i) states that $m(z, w) = 2$, and it follows that

$$(2) \quad \delta(z, w) = \delta(z) - m(z, w) + 2 = \delta(z);$$

consequently,

$$(3) \quad \delta(x) = \delta(a) = \delta(y),$$

as asserted. ■

Now we consider ultrageodetic blocks with at least one nontrivial maximal clique.

6.2 Proposition. Suppose that a vertex v in a nontrivial maximal clique K lies in exactly one other maximal clique L . Then the following statements hold for each vertex $z \in S(K; \text{diam})$:

- i) The vertex z lies in exactly $|K|$ maximal cliques.
- ii) With the possible exception of $M(z, v)$, every maximal clique containing z is trivial.
- iii) The maximal clique $M(z, v)$ is trivial iff every vertex in K lies in exactly two maximal cliques.

Proof. Fix a vertex $z \in S(K; \text{diam}) = R(K, v; \text{diam})$, and observe that $L = M(v, z)$. Consider the sets

$$(4) \quad \begin{aligned} A &:= S(z; 1) \cap S(K; \text{diam} - 1), \\ B &:= S(z; 1) \cap S(K; \text{diam}) \cap M(z, v), \text{ and} \end{aligned}$$

$$C := S(z;1) \cap S(K;\text{diam}) \setminus M(z,v).$$

Obviously, A , B , and C are pairwise disjoint, and their union is $S(z;1)$. For each $s \in A$, the fact that $d(s,K) = \text{diam} - 1$ implies that $s \in R(K,w;\text{diam}-1)$ for some $w \in K$, and thus also that $s = \langle z,w;1 \rangle$ for the same w . According to 5.7, for distinct vertices w_1 and w_2 in K ,

$$(5) \quad R(K,w_1;\text{diam}-1) \cap R(K,w_2;\text{diam}-1) = \emptyset;$$

hence, each $s \in A$ is $\langle z,w;1 \rangle$ for a unique $w \in K$, and we see that $|A| = |K|$. Furthermore, A is an independent set, because, for distinct vertices w_1 and w_2 in K , no vertex in $R(K,w_1;\text{diam}-1)$ has a neighbor in $R(K,w_2;\text{diam}-1)$, by 4.11. The maximal clique $M(z,v)$ consists of $\langle z,v;1 \rangle$, z , and $(m(z,v) - 2)$ other vertices that must be at least as far from v as z is, by 4.3; consequently, $B = M(z,v) \setminus \{z, \langle z,v;1 \rangle\}$. We compute that

$$\begin{aligned} (6) \quad \delta(z) &= |A| + |B| + |C| \\ &= |K| + (m(z,v) - 2) + |C|. \end{aligned}$$

Since $B(v;1) = K \cup L$, we see that

$$\begin{aligned} (7) \quad \delta(v,z) &= \delta(v) - m(z,v) + 2 \\ &= (|K| + |L| - 2) - |L| + 2 \\ &= |K|; \end{aligned}$$

On the other hand, $d(v,z) = \text{diam}$, so we can apply 5.21 to obtain from (6) that

$$\begin{aligned} (8) \quad \delta(v,z) &= \delta(z,v) \\ &= \delta(z) - m(z,v) + 2 \\ &= |K| + |C|; \end{aligned}$$

combining (7) and (8), we see that $|K| = |K| + |C|$, and therefore that $C = \emptyset$. Since A is an independent set, we conclude that z lies in exactly $|K|$ maximal cliques, all but possibly $M(z,v)$ of which are trivial; in other words, i) and ii) hold.

Suppose that $m(z,v) = 2$; it follows from (6) and the fact that $C = \emptyset$ that $\delta(z) = |K|$. Fix a vertex $u \neq v$ in K . By 5.21, the fact that $d(u,z) = \text{diam}$ implies that $\delta(u,z) = \delta(z,u)$; but we established above that $m(z,u) = 2$, so it follows that

$$\begin{aligned} (9) \quad \delta(u,z) &= \delta(z,u) \\ &= \delta(z) - m(z,u) + 2 \\ &= |K|. \end{aligned}$$

The nontriviality of K implies that $\delta(u) = \delta(v)$, by 5.17; as a consequence, it follows from (9) that

$$\begin{aligned} (10) \quad m(u,z) &= \delta(u) - \delta(u,z) + 2 \\ &= \delta(v) - \delta(u,z) + 2 \\ &= (|K| + |L| - 2) - |K| + 2 \\ &= |L|. \end{aligned}$$

Due to the fact that

$$(11) \quad \delta(u) = \delta(v) = |K| + |L| - 2,$$

it must be that K and $M(u,z)$ are the only maximal cliques containing u .

Now suppose that every vertex in K lies in exactly two maximal cliques, and fix a vertex $u \neq v$ in K . We saw above that A is an independent set; in particular, $\langle z, u; 1 \rangle$

and $\langle z, v; 1 \rangle$ are nonadjacent, and $M(z, u) \neq M(z, v)$. According to i), the fact that u lies in exactly one maximal clique other than K implies that, with the possible exception of $M(z, u)$, every maximal clique containing v is trivial; specifically, $M(z, v)$ is trivial. ■

6.3 Corollary. If every maximal clique is nontrivial, then each vertex lies in at least three maximal cliques. ■

Proof. By 2.11, the absence of extreme points implies that every vertex lies in at least two maximal cliques. If every maximal clique is nontrivial, then no vertex lies in exactly two of them, by 6.2. ■

6.4 Proposition. Suppose that a vertex x lies in at least three different maximal cliques. Then, for each $k < \text{diam}$, and each pair of vertices $y, z \in S(x; k)$,

$$r(x, y; k+1) = r(x, z; k+1).$$

Proof. Fix a positive integer $k < \text{diam}$ and vertices $y, z \in S(x; k)$; put $v := \langle x, y; 1 \rangle$ and $w := \langle x, z; 1 \rangle$. By assumption, x has a neighbor u that lies in neither $M(x, v)$ nor $M(x, w)$; set $K = M(x, u)$. Since v and w do not lie in K , it follows from 4.3 that, for each $t \neq x$ in K , $d(t, y)$ and $d(t, z)$ both exceed k . Consequently, $d(K, y) = d(K, z) = k$, and we see that y and z lie in $R(K, x; k)$. According to 4.22, then,

$$(12) \quad R(x, y; k+1) = R(K, y; k+1)$$

and

$$(13) \quad R(x, z; k+1) = R(K, z; k+1).$$

Applying 5.23, we see that $r(K, y; k+1) = r(K, z; k+1)$, and thus that $r(x, y; k+1) = r(x, z; k+1)$. ■

Now we consider ultrageodetic blocks with both trivial and nontrivial cliques; each must contain a vertex lying in both a trivial maximal clique and a nontrivial maximal clique. We show first that such a vertex must lie in exactly two maximal cliques if $\text{diam} > 2$; the latter condition is necessary, as can be seen in Figure 15.

6.5 Proposition. Suppose that $\text{diam} > 2$, and let x be a vertex that lies in both a nontrivial maximal clique K and a trivial maximal clique L . Then x lies in no other maximal clique.

Proof. Suppose the contrary; that is, suppose that x lies in at least three maximal cliques. Let $v \in K$ and $w \in L$ be neighbors of x , and fix vertices $y \in R(x, v; 2)$ and $z \in R(x, w; 2)$. We consider two cases: either $m(w, z) = 2$ or $m(w, z) > 2$.

Case 1. Suppose that $m(w, z) = 2$. Since K and L are not the only maximal cliques containing x , we see that $r(K, x; 1) > 1$. By 5.24, $r(K, x; 1) = r(K, v; 1)$, because K is nontrivial; furthermore, according to 5.12, we have that $R(K, v; 1) = R(x, v; 2)$, and it follows that

$$(14) \quad 1 < r(K, x; 1) = r(x, v; 2).$$

The fact that x lies in at least three maximal cliques permits us to apply 6.4: $r(x, v; 2) = r(x, w; 2)$; since we know from 4.22 that $R(x, w; 2) = R(K, w; 2)$, (14) yields the inequality

$$(15) \quad 1 < r(K, x; 1) = r(x, w; 2) = r(K, w; 2).$$

Now 5.24 states that $r(K, w; 2) = r(K, y; 2)$, and 5.12 implies that $R(K, y; 2) = R(x, y; 3)$, since $\text{diam} \geq 3$; another application of 6.4 shows that $r(x, y; 3) = r(x, z; 3)$, so we infer from (15) that

$$(16) \quad r(K, x; 1) = r(x, z; 3).$$

Since $L = M(x, w)$ and $z \in R(L, w; 1)$, it follows from 5.12 and 4.22 that $R(x, z; 3) = R(L, z; 2) = R(w, z; 2)$; from (16), then, we deduce that

$$(17) \quad r(K, x; 1) = r(w, z; 2).$$

In view of the fact that $R(L, w; 1) = R(x, w; 2)$, which is a consequence of 5.12, (15) implies that $r(L, w; 1) > 1$; hence, the assumption that $m(w, z) = 2$ ensures that w lies in at least three maximal cliques. We apply 6.4 again to see that $r(w, z; 2) = r(w, x; 2)$; by (17), then,

$$(18) \quad r(K, x; 1) = r(w, x; 2).$$

According to 5.20, $r(w, x; 2) = \delta(x, w) - 1$, but the latter quantity is just $\delta(x) - 1$, since $m(w, x) = |L| = 2$. Hence, it follows from (18) that

$$(19) \quad r(K, x; 1) = \delta(x) - 1.$$

Now 5.12 states that $R(K, x; 1) = R(v, x; 2)$, and it follows from 5.20 that $r(v, x; 2) = \delta(x, v) - 1$; we see then that $r(K, x; 1) = \delta(x, v) - 1$. Since $m(x, v) = |K|$, it follows that

$$(20) \quad r(K, x; 1) = \delta(x) - |K| + 1.$$

Combining (19) and (20), we come to the conclusion that $|K| = 2$, which is nonsense.

Case 2. Suppose now that $m(w,z) > 2$; by 5.17, that implies that $\delta(z) = \delta(w)$, and the fact that $M(w,x) = L$ is trivial implies that $\delta(w) = \delta(w,x)$. By 5.20, we know that $\delta(w,x) - 1 = r(x,w;2)$, and we deduce from 4.22 that the reaches $R(x,w;2)$ and $R(K,w;2)$ coincide; furthermore, $r(K,w;2) = r(K,y;2)$, by 5.24, so we see that

$$(21) \quad \delta(z) - 1 = \delta(w,x) - 1 = r(x,w;2) = r(K,y;2).$$

Now 5.12 implies that $r(K,y;2) = r(x,y;3)$, since $\text{diam} \geq 3$; because x lies in at least three maximal cliques, we can apply 6.4: $r(x,y;3) = r(x,z;3)$, and it follows that $r(K,y;2) = r(x,z;3)$. Applying 5.12 and 4.22, we see that $r(x,z;3) = r(L,z;2) = r(w,z;2)$. By 5.20, we have that $r(w,z;2) = \delta(z,w) - 1$, and it follows immediately that

$$(22) \quad \delta(z) - 1 = \delta(z) - m(z,w) + 1.$$

That is, $m(z,w) = 2$, in contradiction to the assumption.

In each case, we reach a contradiction, so it must be that x lies in only two maximal cliques. ■

6.6 Proposition. Suppose that $\text{diam} > 2$ and that there is a trivial maximal clique. Then every vertex lies in a trivial maximal clique.

Proof. Since the graph is connected, it suffices to prove that, for every vertex that lies in a trivial maximal clique, each of its neighbors lies in a trivial maximal clique. Fix a vertex x in a trivial maximal clique K , and consider a neighbor y of x . If $L := M(x,y)$ is trivial, there is nothing to prove; suppose, then, that

$m(x,y) = n > 2$. By 6.5, the fact that $\text{diam} > 2$ ensures that x lies in no maximal clique other than K and L ; that is,

$$(23) \quad \delta(x) = |K| + |L| - 2 = 2 + n - 2 = n.$$

Since L is nontrivial, it follows that $\delta(y) = \delta(x)$, by 5.17; consequently, y has exactly one neighbor outside L , say z , and $M(y,z)$ is trivial. ■

6.7 Corollary. Suppose that there exists a trivial maximal clique, and that $\text{diam} > 2$. Then, for each nontrivial maximal clique K , every vertex in K has degree $|K|$.

Proof. Fix a vertex x in a nontrivial maximal clique. By 6.6, the existence of a trivial maximal clique in the graph implies that x lies in one. Now, the fact that $\text{diam} > 2$ implies that x lies in no other maximal clique, by 6.5. ■

In some ultrageodetic graphs of diameter two, there are vertices lying in exactly two maximal cliques, both of which are nontrivial (see Figure 15). Lee [30] characterized such graphs in terms of families of mutually orthogonal Latin squares.

6.8 Proposition. Suppose that there is a trivial maximal clique, and that $\text{diam} > 2$. Then, for each nontrivial maximal clique K , every vertex in $S(K;\text{diam})$ has degree $|K|$, and lies in no nontrivial maximal clique.

Proof. Let K be a nontrivial maximal clique, and fix vertices $x \in K$ and $a \in S(K;\text{diam})$ (by 5.2, such an a ex-

ists). According to 6.6, x lies in a trivial maximal clique; since K is nontrivial, it follows that $\delta(x) = |K|$, by 6.7. Since x has $|K| - 1$ neighbors in K , we see that $r(K, x; 1) = 1$; that is, $\langle x, a; 1 \rangle$ is the unique neighbor of x outside K , so that $m(x, a) = 2$. It follows from 5.21 that

$$(24) \quad \delta(a, x) = \delta(x, a) = \delta(x) - m(x, a) + 2 = |K|.$$

That same argument holds for each $y \neq x$ in K ; fix such a y , and note that

$$(25) \quad \delta(a, x) = \delta(x) = \delta(y) = \delta(y, a) = \delta(a, y).$$

Since

$$(26) \quad \delta(a, x) = \delta(a) - m(a, x) + 2$$

and

$$(27) \quad \delta(a, y) = \delta(a) - m(a, y) + 2,$$

it follows that $m(a, x) = m(a, y)$. In view of the fact that

$$(28) \quad \langle a, x; 1 \rangle \in R(K, x; \text{diam}-1) \cap M(a, x)$$

and

$$(29) \quad \langle a, y; 1 \rangle \in R(K, y; \text{diam}-1) \cap M(a, y),$$

knowing that $R(K, x; \text{diam}-1) \cap R(K, y; \text{diam}-1) = \emptyset$ (by 5.7) is sufficient for us to deduce from 4.11 that $\langle a, x; 1 \rangle$ and $\langle a, y; 1 \rangle$ are nonadjacent, and thus that $M(a, x) \neq M(a, y)$. Hence, a lies in at least two different maximal cliques with the same number of vertices; by 6.6, a must lie in a trivial maximal clique, so it follows from 6.5 that every maximal clique containing a is trivial. As a result, we see that $\delta(a) = \delta(a, y) + m(a, y) - 2 = |K|$. ■

We turn now to the investigation of suspended paths in ultrageodetic blocks of diameter greater than two.

6.9 Definition. For each vertex x , we define the maximum clique number $m(x)$ to be the largest number of vertices in a maximal clique containing x .

6.10 Proposition. Let K be a nontrivial maximal clique, and suppose that there exists a trivial maximal clique. Then there exists a positive integer k such that, for every vertex $x \in K$, the following statements hold:

- i) There is a suspended path $Q(x,y)$ such that $y \in S(x;k)$ and $\lambda(Q) = k$.
- ii) For integers i,j such that $0 \leq i < j \leq k$, $r(K;i,j) = 1$.
- iii) Every vertex in $S(K;k)$ has the same degree and the same maximum clique number.

Proof. Each vertex outside K lies in a distance-preserving pyramidal subgraph P with base K such that $\text{diam}(P) = \text{diam}(G)$, by 5.2; let a be the apex of such a pyramid. According to 6.8, the existence of a nontrivial maximal clique implies that $\delta(a) = |K|$, since $d(a,K) = \text{diam}$. As K is nontrivial, the initial vertices of the geodesics joining the vertices in K to a all have degree $|K| > 2$; consequently, each such geodesic has a unique decomposition into suspended paths, the first of which contains exactly one vertex in K . Fix distinct vertices x and y in K , and consider suspended paths Q and T whose initial vertices are x and y , respectively; without loss of general-

ity, we suppose that $\lambda(Q) \leq \lambda(T)$. Set $j := \lambda(Q)$ and $k := \lambda(T)$, and label the vertices of Q and T as follows:

$$(30) \quad Q: x = x_0 - x_1 - \dots - x_j.$$

$$T: y = y_0 - y_1 - \dots - y_k.$$

Since x_j and y_j lie on the geodesics joining x to a and y to a , respectively, the fact that $d(K, a) = \text{diam}$ implies that the vertices $x_j = \langle x, a; j \rangle$ and $y_j = \langle y, a; j \rangle$ both lie in $S(K; j)$, by 4.21. Now, as $x_{j-1} = \langle x_j, x; 1 \rangle$ and $y_{j-1} = \langle y_j, y; 1 \rangle$, we see that $M(x_j, x) = M(x_j, x_{j-1})$ and $M(y_j, y) = M(y_j, y_{j-1})$. Furthermore, both of those maximal cliques are trivial: if $j = 1$, then x_j (resp., y_j) is the only neighbor of x (resp., y) outside K , since $\delta(x) = \delta(y) = |K|$, by 6.7; and, if $j > 1$, then x_{j-1} and y_{j-1} , as internal vertices of the suspended paths Q and T , both have degree two. Thus, we have that

$$(31) \quad m(x_j, x) = m(y_j, y) = 2.$$

According to 4.22, since $d(K, x_j) = d(x, x_j) = j$,

$$(32) \quad R(x, x_j; j+1) = R(K, x_j; j+1);$$

likewise, since $d(K, y_j) = d(y, y_j) = j$,

$$(33) \quad R(y, y_j; j+1) = R(K, y_j; j+1).$$

By 5.24, $r(K, x_j; j+1) = r(K, y_j; j+1)$, so it follows from (32) and (33) that

$$(34) \quad r(x, x_j; j+1) = r(y, y_j; j+1).$$

From 5.8, we know that

$$(35) \quad B(x_j; 1) = R(x, x_j; j+1) \cup M(x_j, x);$$

likewise,

$$(36) \quad B(y_j; 1) = R(y, y_j; j+1) \cup M(y_j, y).$$

Counting elements on both sides of (35) and (36), we find that

$$(37) \quad \delta(x_j) + 1 = r(x, x_j; j+1) + m(x_j, x)$$

and

$$(38) \quad \delta(y_j) + 1 = r(y, y_j; j+1) + m(y_j, y).$$

Now we combine (37) and (38) with the aid of (31) and (34) to deduce that

$$(39) \quad \delta(x_j) = \delta(y_j).$$

But x_j is the terminal vertex of the suspended path Q , so $\delta(x_j) \geq 3$; consequently, y_j is the terminal vertex of R , and $j = k$. We see now that statement i) holds.

To show that ii) holds, it suffices to establish that $r(K; j, j+1) = 1$ for nonnegative $j < k$. For $j = 0$, as $\delta(x) = |K|$ (again, by 6.7), the fact that x has $|K| - 1$ neighbors in K implies that $r(K, x; 1) = 1$; by definition, then, $r(K; 0, 1) = 1$. For $j > 0$, x_j is an internal vertex of Q , and therefore has degree two; again, we see that $r(K; j, j+1) = r(K, x_j; j+1) = 1$.

To see that iii) holds, let w be a vertex in $S(K; k)$, and consider a distance-preserving pyramid P with base K such that $w \in V(P)$ and $\text{diam}(P) = \text{diam}(G)$, as provided by 5.2. As w lies on a geodesic joining some vertex in K to the apex b of P , we know from the preceding argument that w is the terminal vertex of a suspended path whose

initial vertex lies in K ; by (39), all such vertices have the same degree. We still need to verify that each two vertices in $S(K;k)$ have the same maximum clique number. Fix vertices $v, w \in S(K;k)$. If neither v nor w lies in a nontrivial maximal clique, then $m(v) = m(w) = 2$. If $k = \text{diam}$, then it follows from 6.8 that $m(v) = m(w) = 2$. Suppose that iii) does not hold. Without loss of generality, then, we may assume that $k < \text{diam}$, and also that $m(w) = \delta(w) = \delta(v) > m(v) = 2$. Fix vertices x and y in K such that $v \in R(K,x;k)$ and $w \in R(K,y;k)$; as we have already shown that ii) holds, it follows that

$$(40) \quad R(K,x;k) = \{v\} \quad \text{and} \quad R(K,y;k) = \{w\}.$$

Now let a be the apex of a distance-preserving pyramidal subgraph H with base K such that $\text{diam}(H) = \text{diam}(G)$; by 5.2, such an H exists. Then

$$(41) \quad d(a,K) = d(a,x) = d(a,y) = \text{diam};$$

consequently, $\langle x,a;k \rangle \in R(K,x;k)$ and $\langle y,a;k \rangle \in R(K,y;k)$, so (40) implies that $v = \langle x,a;k \rangle$ and $w = \langle y,a;k \rangle$. That is, $d(v,a) = d(w,a) = \text{diam} - k$; set $j := \text{diam} - k$. By 6.8, $\delta(a) = |K| > 2$ and $m(a) = 2$; therefore, we can apply 6.4 to obtain the fact that

$$(42) \quad r(a,v;j+1) = r(a,w;j+1).$$

By 5.8, $B(v;1)$ and $B(w;1)$ can be decomposed as follows:

$$(43) \quad B(v;1) = R(a,v;j+1) \cup M(v,a).$$

$$(44) \quad B(w;1) = R(a,w;j+1) \cup M(w,a).$$

By counting elements on both sides of (43) and (44), we find

that

$$(45) \quad \delta(v) + 1 = r(a, v; j+1) + m(v, a),$$

and

$$(46) \quad \delta(w) + 1 = r(a, w; j+1) + m(w, a).$$

By 5.7, there is a unique nontrivial maximal clique containing w , and w has exactly one neighbor outside that maximal clique. As the w, a -geodesic is suspended (by ii)),

$\langle w, x; 1 \rangle$ is that neighbor; therefore, we deduce that

$m(w, a) = m(w) = \delta(w)$. On the other hand, $m(v, a) = m(v) = 2$.

Thus, it follows from (45) and (46) that

$$(47) \quad r(a, v; 1) = \delta(v) - 1 > 1 = r(a, w; 1),$$

in contradiction to (42), so we conclude that iii) holds. ■

6.11 Proposition. Suppose that a vertex v lies in at least three maximal cliques, and that $m(v) = 2$. Then there are $\delta(v)$ suspended geodesics with initial vertex v , all having the same length; moreover, the terminal vertices of those paths all have the same degree and the same maximum clique number.

Proof. We break the proof into seven parts.

Part 1. First we demonstrate that every neighbor of v lies on a suspended geodesic with initial vertex v : for a vertex $w \in S(v; 1)$, let $vP(w, x)$ be a chordless path that is not T -extendible; according to 3.23, $\lambda(vP) = \text{diam}$. Consequently, vP is the unique v, x -path in T , and thus is a geodesic; in particular, $d(v, x) = \text{diam}$. It follows from

5.21 that $\delta(v, x) = \delta(x, v)$; as $\delta(v, x) = \delta(v) - m(v, x) + 2$, the fact that $m(v) = 2$ implies that $\delta(v, x) = \delta(v)$. But $\delta(v) > 2$, so $\delta(x) \geq \delta(x, v) = \delta(v) > 2$. That is, the initial and terminal vertices of vP both have degree greater than two, and it follows that vP has a unique decomposition into suspended subpaths; w clearly lies on the first of those.

Part 2. Now we establish that all suspended paths with initial vertex v have the same length, and that their terminal vertices have the same degree. Let $P(v, w)$ and $Q(v, y)$ be suspended paths such that $\lambda(P) \leq \lambda(Q)$; set $j := \lambda(P)$ and $k := \lambda(Q)$, and label the vertices of P and Q as follows:

$$(48) \quad P: v = w_0 - w_1 - \dots - w_j = w.$$

$$Q: v = y_0 - y_1 - \dots - y_k = y.$$

As we saw above, P and Q are geodesics; consequently, w_j and y_j both lie in $S(x; j)$. Since $w_{j-1} = \langle w_j, v; 1 \rangle$ and $y_{j-1} = \langle y_j, v; 1 \rangle$, we see that $M(w_j, v) = M(w_j, w_{j-1})$ and $M(y_j, v) = M(y_j, y_{j-1})$, by definition. On the one hand, if $j = 1$, then $m(y_j, y_{j-1}) = m(w_j, w_{j-1}) = 2$, since $m(v) = 2$. On the other hand, if $j > 1$, then w_{j-1} and y_{j-1} both have degree two, since they are internal vertices of suspended paths; as w_{j-2} is not adjacent to w_j and y_{j-2} is not adjacent to y_j (P and Q are geodesics), it follows that $m(y_j, y_{j-1}) = m(w_j, w_{j-1}) = 2$. In either case, we obtain the equality

$$(49) \quad m(y_j, v) = m(w_j, v) = 2.$$

Suppose that $j = \text{diam}$; then $j = k$, and P and Q have the same length. According to 5.21, $\delta(v, w) = \delta(w, v)$ and $\delta(v, y) = \delta(y, v)$; moreover, $m(y, v) = m(w, v) = 2$, by (49), and $m(v, y) = m(v, w) = m(v) = 2$, by assumption. Thus we compute that

$$\begin{aligned}
 (50) \quad \delta(y) &= \delta(y, v) + m(y, v) - 2 \\
 &= \delta(v, y) + m(v, y) - 2 \\
 &= \delta(v) \\
 &= \delta(v, w) + m(v, w) - 2 \\
 &= \delta(w, v) + m(w, v) - 2 \\
 &= \delta(w).
 \end{aligned}$$

Now suppose that $j < \text{diam}$; then it follows from 6.4 that

$$(51) \quad r(v, w_j; j+1) = r(v, y_j; j+1).$$

Furthermore, according to 5.8,

$$(52) \quad B(w_j; 1) = R(v, w_j; j+1) \cup M(w_j, v),$$

and

$$(53) \quad B(y_j; 1) = R(v, y_j; j+1) \cup M(w_j, v).$$

By counting elements on each side in (52) and (53), we find that

$$(54) \quad \delta(w_j) + 1 = r(v, w_j; j+1) + m(w_j, v),$$

and

$$(55) \quad \delta(y_j) + 1 = r(v, y_j; j+1) + m(y_j, v).$$

Now (49) and (51) allow us to combine (54) and (55), and we see that $\delta(w_j) = \delta(y_j)$. But w_j is the terminal vertex of the suspended path P , so its degree is at least three. It

follows that y_j is the terminal vertex of Q ; in other words, $\lambda(P) = j = k = \lambda(Q)$; and $\delta(y) = \delta(w)$, as asserted.

Part 3. Let $k \leq \text{diam}$ be the common length of the suspended paths with initial vertex v ; in the remainder of the proof, we verify that every vertex in $S(v;k)$ has the same maximum clique number. To that end, fix a vertex $u \in S(v;k)$ such that $m(u) > 2$. (If no such u exists, then $m(u) = 2$ for every u in $S(v;k)$, and the assertion holds.) Let K be a maximal clique with $m(u)$ vertices such that $u \in K$. We see that $K \neq M(u,v)$, since $m(u,v) = 2$ (by (49)). Now $M(u,v)$ is the unique maximal clique containing u that is closer to v than u is to v , by 4.4; hence, it must be that $d(K,v) = k$. Furthermore, according to 6.7, each vertex in K has degree $|K|$. Let P be the u,v -geodesic; as P is suspended and has no vertex other than u in K , these three facts follow from 6.10:

(56) Every suspended path with initial vertex in K
and terminal vertex outside K has length
 k .

(57) For $i < k$, $r(K;i,k) = 1$.

(58) Every vertex in $S(K;k)$ has degree $\delta(v)$.

Throughout the remainder of the proof, the hypotheses of this part (and thus also their implications) remain in effect.

Part 4. The hypotheses of Part 3 remain in effect; suppose in addition that $k = \text{diam}$. Then it follows from

5.2 that v is the apex of a distance-preserving pyramid P with base K . Proposition 6.9 states that $\delta_G(v) = |K|$; as we already know that each vertex in K has degree $|K|$ in G , and that all of the internal vertices of the $|K|$ geodesics joining v and K have degree 2, it follows that, for every vertex $y \in V(P)$, $\delta_G(y) = \delta_P(y)$. That is, no vertex in $V(P)$ is adjacent to a vertex in $V(G) \setminus V(P)$; hence, since G is connected, it must be that $V(P) = V(G)$. We conclude that G is a pyramid with base K and apex v , and $S(v; k) = K$.

Part 5. Again, the hypotheses of Part 3 remain in effect; suppose further that $\text{diam} > k$. We first establish the following fact:

(59) No two vertices in $S(v; k)$ are adjacent.

Otherwise, the adjacency of $x, y \in S(v; k)$ would give rise to one of the following two contradictions: for $k > 1$, the cycle QTx formed from suspended paths $Q(x, v)$, $T(v, y)$, and $y - x$ would be chordless, since T and Q are geodesics, and its length would be $2k + 1 < 2 \cdot \text{diam} + 1$, in contradiction to the fact that $g^* = 2 \cdot \text{diam} + 1$ (by 5.2); for $k = 1$, the vertices v, x , and y would form a triangle, contrary to the assumption that $m(v) = 2$.

Now we establish this statement:

(60) For vertices $x, y \in S(v; k)$ and suspended paths $Q(x, v)$ and $T(v, y)$, the path QT is a geodesic.

By the preceding argument, QT is chordless. If

$\lambda(QT) < \text{diam}$, then QT belongs to T , and is therefore a

geodesic (since $T = \Gamma$; see 2.7). On the other hand, if $\lambda(QT) \geq \text{diam}$, then there is a vertex z such that $\lambda((QT) \setminus x, z) = \text{diam}$; it follows that the chordless path $(QT) \setminus x, z$ belongs to $T = \Gamma$, and thus that $d(x, z) = \text{diam}$. By 5.21, then, $\delta(x, z) = \delta(z, x)$. As Q and T are suspended paths, the fact that $m(v) = 2$ ensures that the predecessor of z in QT and the successor of x in QT both have maximum clique number two; consequently, $m(x, z) = m(z, x) = 2$, so we see that $\delta(x) = \delta(x, z)$ and $\delta(z) = \delta(z, x)$. Hence, $\delta(x) = \delta(x, z) = \delta(z, x) = \delta(z)$. But x is a terminal vertex of a suspended path, so $\delta(x) > 2$. It follows that z is a terminal vertex of Q ; since $k < \text{diam}$, it must be that $z = y$, and QT is a geodesic. In establishing (60), we also found that

$$(61) \quad \text{diam} \geq 2k.$$

Part 6. This is a continuation of Part 5, all of whose hypotheses still stand. Our goal here is to sharpen (61) by proving that $\text{diam} > 2k$. Recall from Part 3 that the vertex $u \in S(v; k)$ lies in the nontrivial maximal clique K , and fix distinct vertices w and x in $S(v; k) \setminus \{u\}$; by Part 1, that is possible, since $\delta(v) \geq 3$. Consider suspended paths $Q(v, w)$ and $T(v, x)$; $P(u, v)$ was introduced already in Part 3. As we saw in (60) of Part 5, the paths PQ , PT , and $Q'T$ are geodesics, and it follows that the distance between any two of the vertices u , w , and x is $2k$. Consequently,

$$(62) \quad v = \langle u, w; k \rangle = \langle u, x; k \rangle.$$

Since u is the only vertex of P with maximum clique num-

ber greater than two, the vertex

$$(63) \quad u' := \langle u, v; 1 \rangle = \langle u, w; 1 \rangle = \langle u, x; 1 \rangle,$$

which lies on P , must lie outside K ; by 4.3, then, every vertex in K lies at least as far from w , x , and v as u does. It follows that $d(K, w) = d(K, x) = 2k$; by the same reasoning, we proved in Part 3 that $d(K, v) = k$. Fix a vertex $t \neq u$ in K ; by 6.10, the fact that $P(u, v)$ is a suspended path of length k with $v \in S(K; k)$ implies that $r(K; 0, k) = 1$. According to 5.26, then, $r(K, t; k) = 1$; let s be the unique vertex in $R(K, t; k)$. Since

$$(64) \quad R(K, t; k) \cap R(K, u; k) \subseteq R(K, t) \cap R(K, u) \cap S(K; k),$$

it follows from 5.7 that

$$(65) \quad R(K, t; k) \cap R(K, u; k) \subseteq S(K; \text{diam}) \cap S(K; k) = \emptyset.$$

Specifically, $s \neq v$.

Suppose now that $\text{diam} = 2k$; as we shall see, that supposition leads to a contradiction: by 4.3 and (62), v is the unique vertex whose distance from both w and x is k ; however, since $d(t, w) = \text{diam} = d(t, x)$, both $\langle t, w; k \rangle$ and $\langle t, x; k \rangle$ must lie in $R(K, t; k) = \{s\}$, and we deduce that $d(s, w) = k = d(s, x)$, contrary to the uniqueness of v .

Part 7. This is a continuation of Part 6; consequently, the hypotheses and conclusions of Parts 3, 5, and 6 still stand. We want to show now that every vertex in $S(v; k)$ has maximum clique number $m(u)$. As our choice of $w \neq u$ in Part 6 was arbitrary, it will suffice to prove that $m(w) = m(u)$. Furthermore, we need only show that

$m(w) > 2$, for we showed in Part 2 that $\delta(w) = \delta(u) = |K|$, and, if w lies in a nontrivial maximal clique, then it follows from 6.7 that the number of vertices in that clique is $\delta(w)$. Suppose, then, that $m(w) = 2$; we shall obtain a contradiction, thus establishing the result. Because $m(w) = 2$, the fact that $\delta(w) = |K| \geq 3$ implies that Parts 1, 2, 3, and 5 hold for w in place of v . Let $y \neq v$ lie in $S(w;k)$, and consider the suspended path $U(w,y)$. According to Part 5, the path QU is a geodesic, and $d(v,y) = 2k$; it follows that $y \neq u$. Moreover, if y were adjacent to u , then we would have a chordless cycle $C := PQUu$ whose length would satisfy the inequality

$$(66) \quad \lambda(C) = 3k + 1 < 4k + 1 < 2 \cdot \text{diam} + 1 = g^*,$$

which is nonsense; consequently, y and u are not adjacent.

We show now that $\text{diam} \geq 3k$. Suppose, on the contrary, that $\text{diam} < 3k$. Then the chordless path PQU has a subpath $PQUu,z$ of length diam , and z is an internal vertex of U ; as the unique u,z -path in T , $PQUu,z$ is a geodesic. It follows that $d(u,z) = \text{diam}$, and thus that $\delta(u,z) = \delta(z,u)$, by 5.21. Now $\langle u,z;1 \rangle$ is the only neighbor of u outside K , so $m(u,z) = 2$, and it follows that $\delta(z,u) = \delta(u) \geq 3$. But $\delta(z,u) \leq \delta(z) = 2$, and we have a contradiction; consequently, it must be that $\text{diam} \geq 3k$.

Suppose that $k = 1$. Then u is the unique neighbor of v in K , and $M(t,v) = K$. Now, on the one hand, 5.8 states that $B(t;1) = K \cup R(v,t;3)$; on the other hand, the

only neighbor of t outside K is s , by (57) of Part 3. Consequently, $R(v, t; 3) = \{s\}$; since $d(v, t) = d(v, y) = 2$, it follows from 6.4 that

$$(67) \quad r(v, y; 3) = r(v, t; 3) = 1.$$

According to 5.20, $r(v, y; 3) = \delta(y, v) - 1$; due to the fact that $m(y, v) = m(y, w) = m(w) = 2$, $\delta(y, v) = \delta(y)$, so it follows from (67) that $\delta(v) = 2$. Due to the fact that $d(w, y) = d(w, v) = 1$, an application of Part 2 to w in place of v yields the equality $\delta(y) = \delta(v)$, contrary to hypothesis.

Now suppose that $k > 1$. Since $v \in R(K, u; k)$ and $s \in R(K, t; k)$, it follows from 5.7 that v and s are not adjacent. Let $W(t, s)$ be the t, s -geodesic, which is a suspended path. Then the path $X := (P't)W$ is chordless, and $\lambda(X) = 2k + 1 < 3k$, since $k > 1$. Consequently, $Q \in T = \Gamma$, and $d(v, s) = 2k + 1 < \text{diam}$. Now we see that $r(v, s; 2k+2) = \delta(s, v) - 1$, by 5.20. Because $\langle s, v; 1 \rangle$ is an internal vertex of the suspended path W (since $k > 1$), it must be that $m(s, v) = 2$. Hence, $\delta(s, v) = \delta(s)$; by (58) of Part 3, $\delta(s) = \delta(v) > 2$, and we deduce that

$$(68) \quad r(v, s; 2k+2) > 1.$$

Fix $z_1 \in R(v, y; 2k+1)$; we observe that

$$(69) \quad y = \langle z_1, v; 1 \rangle = \langle z_1, w; 1 \rangle,$$

and thus also that

$$(70) \quad M(z_1, v) = M(z_1, w) = M(z_1, y).$$

Since $d(v, z_1) = d(v, s)$, it follows from 6.4 that

$$(71) \quad r(v, s; 2k+2) = r(v, z_1; 2k+2).$$

According to 5.8, $B(z_1; 1) = M(z_1, y) \cup R(y, z_1; 2k+2)$; by the same reasoning, $B(z_1; 1) = M(z_1, w) \cup R(w, z_1; k+2)$. Thus, it follows from (70) that

$$(72) \quad R(w, z_1; k+2) = R(y, z_1; 2k+2).$$

combining (68), (71), and (72), we see that

$$(73) \quad r(w, z_1; k+2) > 1.$$

Now let z_2 be the predecessor of v in P ; we observe that $d(w, z_2) = k + 1$. It follows from 6.4 that

$$(74) \quad r(w, z_2; k+2) = r(w, z_1; k+2);$$

therefore, by (73), we have that

$$(75) \quad r(w, z_2; k+2) > 1.$$

But this is a contradiction, since the fact that $k > 1$ implies that z_2 is an internal vertex of P ; therefore, $\delta(z_2) = 2$, and (75) is incorrect. We have obtained contradictions for $k = 1$ and for $k > 1$, so it cannot be that $m(w) = 2$. ■

6.12 Proposition. Let G be an ultrageodetic block of diameter greater than two with both trivial and nontrivial maximal cliques. Then V can be partitioned into three sets A , B , and C in such a way that the following conditions are satisfied:

- i) $A = \{x \in V; \delta(x) = m(x) > 2\};$
- ii) $B = \{x \in V; \delta(x) > m(x) = 2\};$ and
- iii) $C = \{x \in V; \delta(x) = m(x) = 2\}.$
- iv) A and B are nonempty, and all of the vertices

- in $A \cup B$ have the same degree Δ .
- v) A is a disjoint union of nontrivial maximal cliques, all of which have the same number of vertices ω , and $\omega = \Delta$.
 - vi) There is a positive integer $s \leq \text{diam}$ such that each vertex in A (resp., B) is joined to exactly 1 vertex (resp., $\delta(x)$ vertices) in B (resp., A) by a suspended path of length s , all of whose internal vertices belong to C .
 - vii) Every vertex in C lies on a suspended path joining a vertex in A to a vertex in B .

Proof. We begin by finding a maximal clique L and a positive integer $s \leq \text{diam}$ such that the following conditions are satisfied:

- (76) For every nonnegative integer $k < s$, each vertex v in $S(L; k)$ has degree 2;

and

- (77) For each pair of vertices $x, y \in S(L; s)$,
 $\delta(x) = \delta(y) > 2$ and $m(x) = m(y) = 2$.

To that end, let K be a nontrivial maximal clique, and fix $x \in K$. By 5.2, the sphere $S(K; \text{diam})$ is nonempty; moreover, by 6.9, each vertex in it has degree $|K|$ and lies in no nontrivial maximal cliques; also, each vertex in K has degree $|K|$ and maximum clique number $|K|$. We define the integer i as follows:

- (78) $i := \max\{k; \exists z \in S(K; k), m(z) > 2\}$.

Perhaps $i = 0$; in light of the preceding observations,
 $i < \text{diam}$. Fix a vertex $v \in S(K; i)$ such that $m(v) > 2$,
 and let a be the apex of a distance-preserving pyramidal
 subgraph P of G with base K such that $v \in V(P)$ and
 $\text{diam}(P) = \text{diam}(G)$, as provided by 5.2. Since $m(a) = 2$ (by
 6.8), we see that $a \neq v$. Let $x \in K$ be such that
 $d(x, v) = d(K, v)$; then $v = \langle x, a; i \rangle$. For each nonnegative
 $k \leq \text{diam}$, put

$$(79) \quad x_k := \langle x, a; k \rangle,$$

and define the integers j and s as follows:

$$(80) \quad j := \min\{k > i; \delta(x_k) > 2\}.$$

$$(81) \quad s := j - i$$

Since $\delta(a) > 2$, we have that $0 \leq i < j \leq \text{diam}$. By
 (79), there is a nontrivial maximal clique, say L , contain-
 ing x_i ; according to 6.7, x_i lies in a trivial maximal
 clique, and it follows from 6.6 that x_i lies in exactly
 two maximal cliques. Now (78) yields the fact that
 $m(x_{i+1}) = 2$ and it follows that x_{i+1} is the unique neigh-
 bor of x_i outside L ; since $x_{i+1} = \langle x_i, a; 1 \rangle$, it must be
 that $a \in R(L, x_i)$. Hence, it follows from 4.20 that, for
 each positive $k \leq \text{diam} - i$, $d(L, x_{i+k}) = k$, since
 $x_{i+k} = \langle x_i, a; k \rangle$. But $\delta(x_{i+k}) = 2$ for each $k < s$, and
 $\delta(x_{i+k}) = \delta(x_j) > 2$; consequently, $R(L, x_{i+k-1}) = \{x_{i+k}\}$ for
 $1 \leq k \leq s$, and $r(L, x_{i+s}; s+1) > 1$. According to 6.10, then,
 L and s satisfy (76) and (77).

Now fix s , and let the set A be the union of all
 nontrivial maximal cliques L such that L and s satisfy

(76) and (77); define the set B to consist of all initial vertices of suspended paths that terminate in A , and let C be the set of internal vertices of such paths. According to Part 1, A and B are nonempty. For each maximal clique $L \subseteq A$, and each vertex $w \in S(L;s)$, (75) and (76) ensure that w is the initial vertex of a suspended path terminating in L ; by 6.11, then, every path of length s with initial vertex w is a suspended path whose terminal vertex lies in a nontrivial maximal clique. By 6.10, each such clique satisfies (66) and (67), and hence is a subset of A . That is, for each vertex x in B , $S(x;1) \subseteq A \cup C$; and, for each maximal clique $L \subseteq A$, and every vertex $x \in L$, the unique neighbor of x outside L lies either in B or in C (accordingly as $s = 1$ or $s > 1$), so that $S(x;1) \subseteq A \cup B \cup C$. Let H be the subgraph of G induced by $A \cup B \cup C$. We see that no edge of G joins a vertex $x \in V(H)$ with a vertex $y \in V(G) \setminus V(H)$; since G is connected, it must be that $H = G$. Now we have a partition $V = A \cup B \cup C$ that satisfies i), ii), iii), vi), and vii). As we know from the first part of the proof that A and B are nonempty, all that remains for the verification of iv) is to show that all vertices in $A \cup B$ have the same degree. We prove first that all vertices in B have the same degree. For each geodesic $Q(x,y)$ joining vertices in B , the fact that the degrees of x and y are both greater than two (by ii)) implies that Q has a unique decomposition into suspended paths, say $Q = Q_1 Q_2 \dots Q_m$. For each

such Q , set $z_0 := x$ and $z_m := y$; further, for each positive $i < m$, let z_i be the terminal (resp., initial) vertex of Q_i (resp., Q_{i+1}). Then z_1 must lie in some maximal clique $L \in A$ (by v_1); as Q_1 is the only suspended path with initial vertex z_1 having no other vertex in L (again by v_1), it must be that $z_2 \in L$. Since Q is a geodesic, it cannot be that $z_3 \in L$, so Q_3 is a suspended path of length s whose terminal vertex z_3 lies in B . It follows that $d(x, y) = \lambda(Q) \geq 2s + 1$. Since z_0 and z_3 both lie in $S(L; s)$, it follows from 6.10 that $\delta(z_0) = \delta(z_3)$. Now we prove by induction on $n := d(x, y)$ that $\delta(x) = \delta(y)$. We have just seen that $n \geq 2k + 1$, and that $\delta(x) = \delta(y)$ if $n = 2k + 1$. Suppose now that $n > 2k + 1$, and that $\delta(x) = \delta(y)$ for all vertices $x, y \in B$ such that $d(x, y) < n$; suppose further that $d(x, y) = n$ and that x and y lie in B . Labelling the geodesic $Q(x, y)$ as above, we see that $\delta(x) = \delta(x_3)$ and $\delta(x_3) = \delta(y)$, by the induction hypothesis. Consequently, $\delta(x) = \delta(y)$, and the induction is complete. Now we verify that every maximal clique in A has the same number of vertices, and that every vertex in $A \cup B$ has the same degree: Let Δ be the common degree of the vertices in B , and fix a vertex x in a maximal clique $L \in A$. By 5.2, there is a distance-preserving pyramidal subgraph P of G with base K such that $\text{diam}(P) = \text{diam}(G)$; let a be the apex of such a pyramid. By 6.8, $\delta(a) = |K| > 2$ and $m(a) = 2$. It follows that $a \in B$, and therefore that $|K| = \delta(a) = \Delta$; moreover,

according to 6.7, $\delta(x) = |K| = \Delta$. ■

The decomposition $V = A \cup B \cup C$ provided by 6.12 is depicted in Figure 16 for an ultrageodetic graph with diameter 4.

At this point, we temporarily suspend Hypothesis 5.5 in order to extend the Stemple-Kantor classification 1.7 to ultrageodetic graphs of arbitrary diameter:

6.13 Proposition. The following statements hold for every two-connected ultrageodetic graph with maximum degree Δ and minimum degree δ in which the largest clique has ω vertices:

- i) Every nontrivial maximal clique is a maximum clique.
- ii) Every vertex has degree Δ or δ .
- iii) If $\Delta \neq \delta$, then $\delta = \Delta - \omega + 2$.
- iv) Every vertex in a maximum clique has degree Δ .
- v) Every vertex at distance diam from a nontrivial maximal clique has degree Δ .

Proof. The statements obviously hold if $\text{diam} = 1$ or $\omega = 2$, and it follows from 1.7 that they hold if $\text{diam} = 2$. Suppose that $\text{diam} > 2$ and $\omega > 2$. If every maximal clique is nontrivial, then each adjacent pair of vertices lie in a nontrivial maximal clique, and therefore have the same degree, by 5.17; it follows that the graph is regular. To show that every maximal clique has the same number of vertices, it is sufficient to consider two maximal cliques with

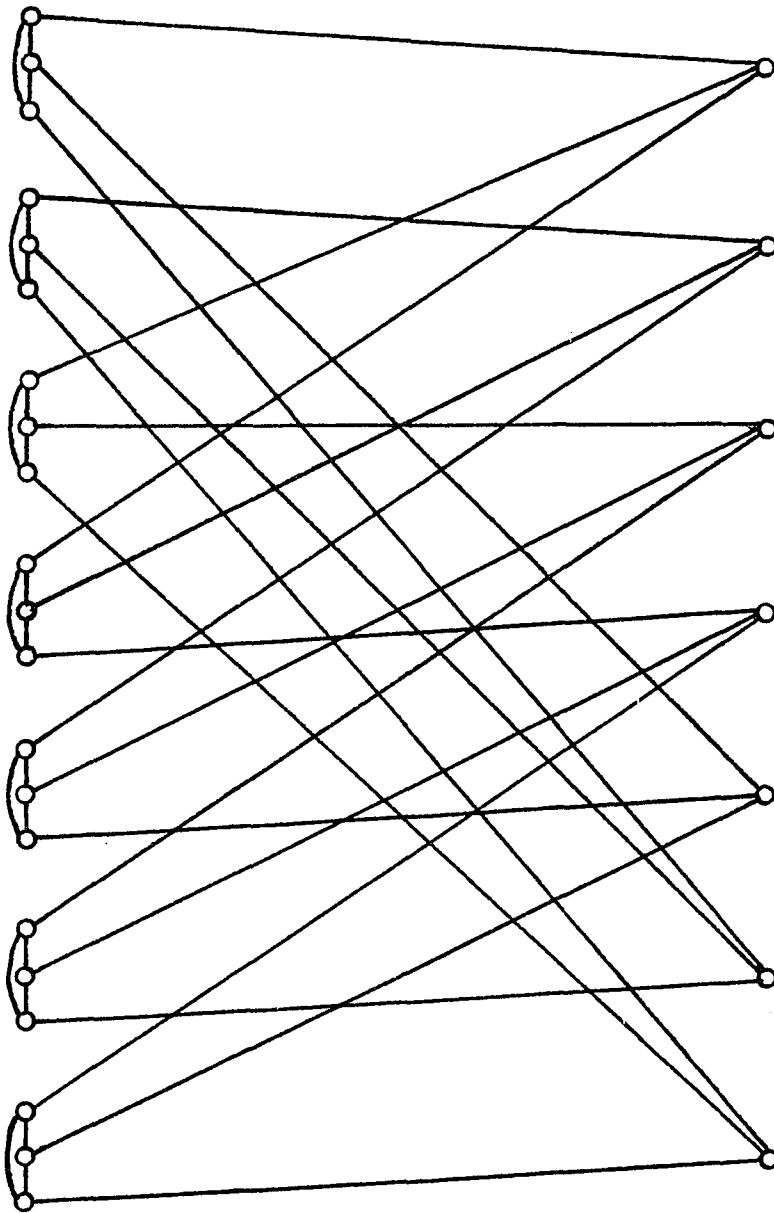


Figure 16.

An ultrageodetic graph of diameter 4 decomposed into the sets A (the vertices on the left) and B (the vertices on the right). In this example, $C = \emptyset$.

nonempty intersection. Suppose, then, that K and L are (nontrivial) maximal cliques with $x \in K \cap L$, and fix vertices $y \in K$ and $z \in L$ distinct from x . According to 6.3, x lies in at least three maximal cliques, so 6.4 is applicable:

$$(82) \quad r(x, y; 2) = r(x, z; 2).$$

By 5.8,

$$(83) \quad B(y; 1) = R(x, y; 2) \cup K;$$

likewise,

$$(84) \quad B(z; 1) = R(x, z; 2) \cup L.$$

By counting elements on both sides of (83) and (84), and applying (82), we see that

$$(85) \quad \delta(y) - |K| = \delta(z) - |L|;$$

as $\delta(y) = \delta(z)$, it follows that $|K| = |L|$. Thus, for graphs in which every maximal clique is nontrivial, statements i), ii), iv), and v) follow, and iii) holds vacuously. For ultrageodetic graphs with both trivial and nontrivial maximal cliques, statements i) - iv) follow from 6.12, and v) is a consequence of 6.8. ■

We now reinstate Hypothesis 5.5.

In order to give an explicit construction of ultrageodetic blocks of diameter greater than two, we apply the concepts and results from finite geometry summarized below; the reader is referred to Dembowski's monograph on finite geometry [14] and Fuglister's first paper on Moore geometries [15] for more detail; except as otherwise noted, the termi-

nology is that of Dembowski ([14], pp. 1-2, 300-305).

A geometry (or incidence structure; cf [14], p. 1) is a triple (P, L, F) consisting of arbitrary finite sets P and L , whose elements are called points and lines (or blocks), respectively, and an incidence relation $F \subseteq P \times L$, whose elements are called flags. The members of $P \cup L$ are called elements of the geometry, and two elements x and y are incident if $(x, y) \in F$ or $(y, x) \in F$. A chain of length n is a finite sequence x_0, x_1, \dots, x_n of elements such that consecutive elements are incident (cf [14], p. 300) ; we say that the chain joins x_0 and x_n . We assume that every two points in a geometry are joined by a chain. The distance $\rho(x, y)$ between two elements of the geometry is the shortest length of a chain joining them. The length of a chain is even iff the first and last elements are both points or both lines. (Since Fuglister [15] considers only chains ("paths") joining two points, the length of a chain in his terminology is one-half of the length as defined here.) Following Fuglister [15], we call a chain irreducible if its points and lines are all distinct (Dembowski's [15] definition of irreducibility is much weaker). Clearly, every chain joining two distinct elements has an irreducible subchain joining those elements. Three kinds of geometries are of particular interest to us here:

A Moore geometry [15] with parameters (a, b) and diameter d is a geometry (P, L, F) that satisfies the follow-

ing three axioms (cf p. 13, 17-19):

- (86) For each $p \in P$, $|\{l \in L; (p,l) \in F\}| = a + 1$.
- (87) For each $l \in L$, $|\{p \in P; (p,l) \in F\}| = b + 1$.
- (88) For distinct $p,q \in P$, there is a unique irreducible chain of length not exceeding $2d$ joining p and q , where $2d = \max\{\rho(p,q); p,q \in P\}$.

A generalized n-gon ([14], p. 301) is a geometry (P,L,F) that satisfies these three axioms:

- (89) For $x,y \in P \cup L$, $\rho(x,y) \leq n$.
- (90) If $\rho(x,y) < n$, then there is a unique chain of length $\rho(x,y)$ joining x and y .
- (91) For each $x \in P \cup L$, there exists a $y \in P \cup L$ such that $\rho(x,y) = n$.

Finally, a finite projective plane ([14], p. 115) is a geometry (P,L,F) that satisfies these three axioms:

- (92) For distinct $p,q \in P$, there is a unique $l \in L$ such that $(p,l) \in F$ and $(q,l) \in F$.
- (93) For distinct $l,m \in L$, there is a unique $p \in P$ such that $(p,l) \in F$ and $(p,m) \in F$.
- (94) There exist four points, no three of which are incident with the same line.

Two observations of Fuglister [15] concerning Moore geometries are of particular importance for us:

- (95) The projective planes are exactly the Moore geometries with parameters (a,a) and diameter $d = 1$.
- (96) Every Moore geometry with parameters (a,a) and

diameter d is a generalized $(2d + 1)$ -gon.

The fundamental (negative) result on the existence of generalized n -gons is the Feit-Higman Theorem (see [15], p. 302); we need only one part of that result:

- (97) For odd $n > 4$, the only generalized n -gons are the ordinary polygons with n points and n lines.

We observe that the following statement holds:

- (98) For two points in a Moore geometry, generalized n -gon, or finite projective plane, there is at most one line incident with both points.

Now we give two constructions of graphs from geometries; the first is implicit in the work of Bose and Dowling on Moore geometries [8].

6.14 Definition. The adjacency graph of a geometry (P, L, F) is the graph with vertex-set P in which two vertices are adjacent if there is a line of the geometry incident with both of them.

6.15 Remark. The set of all points incident with a particular line in a geometry forms a clique in the adjacency graph. In some cases, the adjacency graph does not respect the structure of the geometry; for example, the adjacency graph of a finite projective plane is obviously complete. However, if the geometry satisfies the following condition, then the correspondence is invertible:

- (99) If there are lines incident with each two of the

points p , q , and r , then there is a line incident with all of them.

For, if (99) holds, then every maximal clique in the adjacency graph is the set of points incident with some line of the geometry. By (88), every Moore geometry satisfies (99), so we may regard Moore geometries as being a certain kind of (adjacency) graph. An irreducible chain joining two points in a geometry corresponds to a path in the adjacency graph that is just the subsequence of all points the chain. Conversely, each chordless path in the adjacency graph of a geometry satisfying (99) obviously corresponds to an irreducible chain in the geometry. The length of a chain is twice the length of the corresponding path (if there is one).

In order to describe Moore geometries in graph-theoretic terms, we exploit the correspondence between irreducible chains and locally chordless paths:

6.16 Lemma. For an irreducible chain joining two points in a Moore geometry, the corresponding path in the adjacency graph is locally chordless.

Proof. Suppose the contrary, and let p , q , and r be consecutive vertices in the path such that p and r are adjacent. By (99), there is a line m that is incident with all three points; according to (88), it is the unique line incident with both p and q and also the unique line incident with both q and r . Consequently, the original chain has the subchain p, m, q, m, r , in contradiction to the assumption of irreducibility. ■

In the next result, we verify statements (20) - (22) of Chapter I:

6.17 Proposition. A graph G is isomorphic to the adjacency graph of a Moore geometry with parameters (a,b) and diameter $d > 1$ iff the following three conditions are satisfied:

- (100) Every vertex has degree $a + 1$.
- (101) Every maximal clique has $b + 1$ vertices.
- (102) There is a unique chordless path of length not exceeding d joining each two vertices.

Proof. Suppose first that G is a graph satisfying (100), (101), and (102); then $d \geq \text{diam}$, and G is T_d -geodetic. It follows from 3.21 that G is an ultrageodetic graph of diameter d . Now define the sets P , L , and F as follows:

- (103) $P := V$.
- $L := \{M \subseteq V; M \text{ is a maximal clique}\}.$
- $F := \{(p,l); p \in P \cap l, l \in L\}.$

then G is isomorphic to the adjacency graph of the geometry (P,L,F) , which obviously satisfies (86) and (87).

Moreover, for each irreducible chain C in (P,L,F) , the subsequence of points of C is a locally chordless path P_C in G . If the length of C is no greater than $2d$, then $\lambda(P_C) \leq \text{diam}$, and we can argue as follows that P_C must be chordless: Suppose that P_C has a chord; let x be the first vertex on P_C that is incident with a chord, and let y be the first vertex on P_C after the successor of x such that y and x are adjacent. Then $(P_C \setminus \{x,y\}) \cup \{x,y\}$ is a

locally chordless cycle of length no greater than $d + 1$, in contradiction to the fact that $g^* = 2d + 1$ (by 5.2).

Thus we see that every irreducible chain of length no greater than $2d$ in (P, L, F) corresponds to a member of T in G , so that (88) follows from (102); that is, (P, L, F) is a Moore geometry.

Now suppose that (P, L, F) is a Moore geometry, and let G be its adjacency graph. Then it follows from (88) that each two adjacent vertices in G lie in a unique maximal clique. By 1.5, then, each punctured neighborhood in G is a vertex-disjoint union of cliques; according to (86) and (87), the number of those cliques and the number of vertices in each is constant, so (100) and (101) hold. Since each chordless path of length no greater than d corresponds to an irreducible chain of length no greater than $2d$, (102) follows from (88). ■

With the aid of the preceding result, we can identify the Moore geometries:

6.18 Proposition. The following conditions are equivalent:

- i) Every maximal clique is nontrivial.
- ii) Every maximal clique has the same number $\omega > 2$ of vertices.
- iii) G is the adjacency graph of a thick Moore geometry.

Proof. i) \Rightarrow iii). Suppose that every maximal clique is nontrivial. By 5.17, then, neighbors x and y must have the same degree, and it follows that G is regular. Ac-

cording to 6.3, the fact that every maximal clique is non-trivial implies that every vertex lies in at least three maximal cliques, and 6.4 is applicable: Fix a vertex x , and let y and z be nonadjacent neighbors of x . Set $K := M(x,y) = M(y,x)$ and $L := M(x,z) = M(z,x)$. On the one hand, by 5.20, we have that

$$(104) \quad \delta(y,x) - 1 = r(x,y;2)$$

and

$$(105) \quad \delta(z,x) - 1 = r(x,z;2);$$

on the other hand, we know by 6.4 that $r(x,y;2) = r(x,z;2)$. Consequently, $\delta(y,x) = \delta(z,x)$, and the regularity of G implies that $m(y,x) = m(y,z)$; that is, $|K| = |L|$. It follows that all maximal cliques containing x have the same number of vertices, and thus that every maximal clique has the same number of vertices. Since there is no K_4 -e, every punctured neighborhood is a disjoint union of cliques, each containing the same number of vertices; because G is regular, it follows that every vertex lies in the same number of maximal cliques. By 6.17, then, G is the adjacency graph of a Moore geometry.

iii) \Rightarrow ii). By (21) of Chapter I, in a Moore geometry, every maximal clique contains the same number of vertices.

ii) \Rightarrow i). This is a triviality. ■

6.19 Definition. For a geometry (P,L,F) , we define the point-flag graph $G_1^*(P,L,F)$ to be the graph with vertex-set $V := P \cup F$ whose edges are the unordered pairs of the fol-

lowing two types: $\{p, \langle p, l \rangle\}$ and $\{\langle p, l \rangle, \langle q, l \rangle\}$, with p q in P , l in L , and $\langle p, l \rangle$ and $\langle q, l \rangle$ in F . For $k > 1$, $G_k^*(P, L, F)$ is the graph obtained from $G_1^*(P, L, F)$ via subdivision of every edge of the form $\{p, \langle p, l \rangle\}$ with $k - 1$ new vertices.

6.20 Proposition. Let G be an ultrageodetic graph of diameter greater than two in which there are both trivial and nontrivial maximal cliques. Then either G is a pyramid, or $G \cong G_k^*(P, L, F)$ for some Moore geometry (P, L, F) with parameters (a, b) such that $a = b$.

Proof. We observe first that the hypotheses of 6.12 are satisfied; let $V = A \cup B \cup C$ be the partition given there, let s be the common length of the suspended paths joining vertices in A to vertices in B , and let $a + 1$ be the common degree of vertices in $A \cup B$.

Suppose that $s = \text{diam}$, let $K \subseteq A$ be a maximal clique, and let a be a vertex in B . Since $s = \text{diam}$, $d(K, a) = \text{diam}$, and a is the apex of a distance-preserving pyramidal subgraph H with base K , by 5.2. Moreover, all of the $|K|$ (by iv) and v) of 6.12) neighbors of a in G lie in $V(H) \cap (A \cup C)$, and $S(K; 1)$ is a subset of $V(H) \cap (B \cup C)$. Consequently, there is no edge joining a vertex in $V(H)$ with a vertex in $V(G) \setminus V(H)$, so the fact that G is connected implies that $V(G) = V(H)$, and G is a pyramid.

Now suppose that $s < \text{diam}$; we construct a geometry (P, L, F) as follows: $P := B$, and L is the set of maximal cliques $K \subseteq A$. Finally, F consists of the pairs $\langle p, l \rangle$

such that the vertex $p \in P$ is joined by a suspended path in G to some vertex in the maximal clique $l \in L$. Now $G \cong G_S^*(P, L, F)$, so we want to show that (P, L, F) is a Moore geometry with parameters (a, a) . As conditions (86) and (87) follow immediately from 6.12, it remains only to verify that (88) holds for some d . For each locally chordless path $Q(x, y)$ in G joining vertices $x, y \in B$, it follows from 6.12 that $\delta(x) = \delta(y) \geq 3$, so Q has a unique decomposition into suspended paths; furthermore, each of those paths is of one of the following three types: i) a suspended path of length s joining a vertex in B to a vertex in A ; ii) a suspended path of length one joining two vertices in A ; and iii) a suspended path of length s joining a vertex in A to a vertex in B . Finally, we observe that a suspended path of type i) (resp., ii), iii)) must be followed in Q by a suspended path of type ii) (resp., iii), i)). For each such path $Q(x, y)$, we label the suspended paths and their initial and terminal vertices as follows:

$$(106) \quad Q =: T_1 U_1 V_1 T_2 U_2 V_2 \cdot \cdot \cdot T_n U_n V_n;$$

$$T_i = T_i(x_{i-1}, y_{i-1}); \quad U_i = U_i(y_{i-1}, z_{i-1});$$

$$V_i = V_i(z_{i-1}, x_i).$$

Then $x_0 = x$ and $x_n = y$; furthermore, $x_i \in B$, and $y_i, z_i \in A$ for each i , so T_i is of type i), U_i is of type ii), and V_i is of type iii). Since $\lambda(T_i) = \lambda(V_i) = s$ and $\lambda(U_i) = 1$ for each i , we see that $2s + 1$ must divide $\lambda(Q)$; in particular, $2s+1$ divides $d(x, y)$. For all $p, q \in B$, set $n(p, q) := d(p, q)/(2s+1)$,

and define $n^* := \max\{n(p,q); p,q \in P\}$; we note that $n^* \cdot (2s+1) = \text{diam}$. Now we show that, for $p,q \in P$, the p,q -geodesic $Q(p,q)$ gives rise to an irreducible chain of length $2n(p,q)$ in (P,L,F) : set $x_0 := p$, $n := n(p,q)$, and $x_n := q$, and label Q as before. Let $L_i \in A$ be the maximal clique containing y_i and z_i for $1 \leq i \leq n$. Then we get the chain $x_0, L_1, x_1, \dots, L_n, x_n$. Since Q is a geodesic, the chain is irreducible.

It remains only to show that there is no other irreducible chain of length no more than $2n^*$ in (P,L,F) . Suppose that two points $p,q \in P$ are joined by two different irreducible chains C_1 and C_2 of length no more than n^* , and label C_1 as follows:

$$(107) \quad C_1: x_0, J_1, x_1, \dots, J_j, x_j.$$

Of course, $x_0 = p$ and $x_j = q$; furthermore, $j \leq n^*$. Consider the suspended paths $T_i(x_{i-1}, y_{i-1})$ from x_{i-1} to J_i , $V_i(z_{i-1}, x_i)$ from J_i to x_i , and $U_i(y_{i-1}, z_{i-1})$ in J_i . The path $Q_1 := T_1 U_1 V_1 \dots T_j U_j V_j$ is chordless and has length $\lambda(Q_1) = j(2s+1) \leq n^*(2s+1) = \text{diam}$; that is, $Q_1 \in T$. By applying the same construction to C_2 , we get another chordless path $Q_2 \in T$, in contradiction to the fact that G is ultrageodetic. It follows that (88) holds, and (P,L,F) is a Moore geometry with diameter n^* . ■

By (95) - (97), a thick Moore geometry with equal parameters has diameter one and is a finite projective plane. For those Moore geometries, we prove the converse of 6.20; an example of an ultrageodetic point-flag graph is given in

Figure 17.

6.21 Proposition. Let (P, L, F) be a finite projective plane, and fix a positive integer k . Then $G = G_k^*(P, L, F)$ is an ultrageodetic graph with diameter $3k + 1$.

Proof. As every maximal clique in G is either a non-trivial maximal clique of the form $\{(p, l) \in F; p \in P\}$ for some fixed $l \in L$, or one of the edges in the subdivision of an edge of the form $\{p, (p, l)\}$, the intersection of two maximal cliques in G contains at most one vertex; consequently, G has no subgraph isomorphic to $K_4 - e$, by 1.5. We shall prove that $g^*(G)$ is odd, and that every two vertices of G lie on a chordless cycle of length g^* . It then follows from 5.3 that G is ultrageodetic. Let n be the common degree (by (86), (87), and (95)) of the vertices in $P \cup F$.

We begin by computing a lower bound for g^* . Put $A := P$, let B consist of the vertices of the form (p, l) , and let C be the set of internal vertices of suspended paths; it follows from the definition of $G_k^*(P, L, F)$ that $V = A \cup B \cup C$; furthermore, each vertex in B (resp., A) is adjacent to exactly n (resp., 1) vertices outside B (resp., A). Let Y be a chordless cycle in G ; it follows from the preceding comments that Y must include two suspended paths $V_1(z_0, x_1)$ and $T_2(x_1, y_1)$, where $y_1, z_0 \in A$ and $x_1 \in B$. The other neighbors of y_1 and z_0 in $V(Y) \setminus (V(V_1) \cup V(T_2))$ must lie in A ; let $U_2(y_1, z_1)$ and $U_1(y_0, z_0)$ be suspended paths with $y_0, z_1 \in V(Y) \cap A$. Now

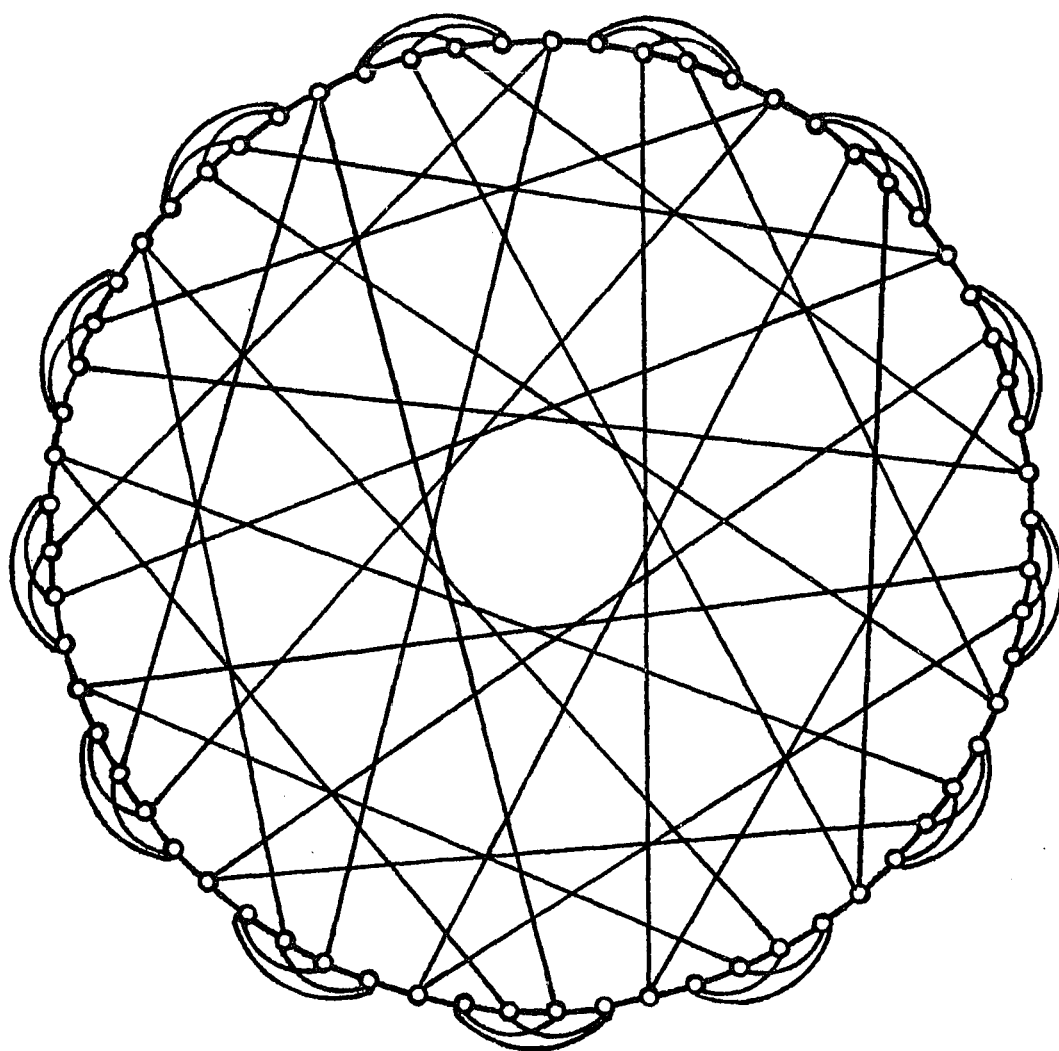


Figure 17.

An ultrageodetic graph of diameter four:
the point-flag graph of the finite
projective plane of order three.

y_0 (resp., z_1) must be joined to another vertex in $B \cap V(Y)$ by a suspended path, say $T_1(x_0, y_0)$ (resp., $V_2(z_1, x_2)$). Furthermore, x_0 and x_2 must be distinct, because the lines l_1 and l_2 of (P, L, F) represented by y_0 and z_0 , and by y_1 and z_1 , respectively, are both incident with x_1 , and thus cannot both be incident with x_0 , by (93). Hence, there must exist vertices $y_2, z_2 \in A$ and suspended paths $T_3(x_2, y_2)$ and $V_3(z_2, x_0)$ lying in Y . It follows that $\lambda(Y) \geq \lambda(V_3 T_1 U_1 V_1 T_2 U_2 V_2 T_3) = 6k + 2$. In particular, $g^* \geq 6k + 3$.

Now we verify that every two vertices lie on a chordless $(6k + 3)$ -cycle. Since every vertex lies on a suspended path from a vertex in A to a vertex in B , it suffices to prove that every pair of those suspended paths lies on a $(6k+3)$ -cycle. Let K_1 and K_2 be maximal cliques in A containing the initial vertices of two suspended paths from A to B , and let q_1 and q_2 be the terminal vertices of those paths. (Perhaps $q_1 = q_2$ or $K_1 = K_2$.) We shall find lines l_1, l_2 , and l_3 in L and points p_1, p_2, p_3 in P such that the following three conditions are satisfied:

- (108) The lines represented by the cliques K_1 and K_2 are among l_1, l_2 , and l_3 .
- (109) The points q_1 and q_2 are among p_1, p_2 , and p_3 .
- (110) The pairs (p_1, l_1) , (p_1, l_2) , (p_2, l_2) , (p_2, l_3) , (p_3, l_3) , and (p_3, l_1) are all flags.

If $q_1 = q_2$, then $K_1 \neq K_2$, since the suspended paths

are distinct. In that case, let l_1 and l_2 be the lines represented by K_1 and K_2 , respectively, and put $p_1 := q_1$; choose arbitrary points $p_3 \neq p_1$ incident with l_1 and $p_2 \neq p_1$ incident with l_2 , and let l_3 be the unique line incident with both p_2 and p_3 .

If $q_1 \neq q_2$, but $K_1 = K_2$, then let l_1 be the line represented by K_1 , and set $p_3 = q_1$ and $p_1 = q_2$. Now let $l_3 \neq l_1$ and $l_2 \neq l_1$ be lines that are incident with p_3 and p_1 , respectively, and let p_2 be the unique point incident with both l_2 and l_3 .

If q_1, q_2, K_1 , and K_2 are all distinct, then let l_1 and l_2 be the lines represented by K_1 and K_2 , respectively. If q_1 is not incident with l_2 and q_2 is not incident with l_1 , then set $p_3 = q_1$ and $p_2 = q_2$; let p_1 be the unique point incident with both l_1 and l_2 , and let l_3 be the unique line incident with both p_2 and p_3 . On the other hand, if one of the given points is incident with both lines, then we may assume without loss of generality that q_2 is incident with l_1 . In that case, put $p_3 = q_1$ and $p_1 = q_2$, and let $p_2 \neq p_1$ be a point incident with l_2 ; finally, let l_3 be the unique line incident with both p_2 and p_3 .

In each case, we see that (108), (109), and (110) hold. Now let L_1, L_2 , and L_3 be the cliques representing l_1, l_2 , and l_3 , respectively, in G . Then the six suspended paths of length k determined by p_1, p_2, p_3, L_1, L_2 , and L_3 , together with the three suspended paths of length one

that join their endpoints (one in each of L_1 , L_2 , and L_3), form a chordless cycle of length $6k + 3$. It follows that $g^* = 6k + 3$, and therefore that G is ultrageodetic, by 5.3; according to 5.2, then, $\text{diam} = 3k + 1$. ■

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Curriculum Vita

Michael F. Bridgland was born on 24 February 1955 in Newport News, Virginia. He received his primary and secondary education in various public schools throughout the United States, and received the High School Diploma from Colonial High School in Orlando, Florida, in June 1972. In September 1972, he entered Florida Technological University, in Orlando, where he studied mathematics and music; he received the degree of Bachelor of Science in Mathematics from that university in June 1977.

On 16 July 1977, he married Donna Momyer.

In August 1977, he began graduate study in mathematics at Louisiana State University, where he was awarded the degree of Master of Science in Mathematics on 3 August 1979.

From August 1979, until July 1980, he studied in West Germany with the aid of a fellowship from the Deutscher Akademischer Austauschdienst. During the months August and September 1979, he studied German at the Goethe-Institut in Freiburg im Breisgau. During the period from October 1979, until July 1980, he was a guest of Professor D. Kölzow at the mathematical institute of Universität Erlangen-Nürnberg.

In July 1980, he moved to Konstanz, where he began work on the translation from German to English of Holomorphic Functions of Several Variables, a textbook by L. Kaup

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He continued his study of Mathematics at Louisiana State University in January 1981. Since August, 1982 he has been a full-time instructor in the Department of Computer Science at Louisiana State University.

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Title of Thesis: Geodetic Graphs and Convexity

Approved:

KENNETH BROOKS REID

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28 JULY 1983